INCOME PROCESS, PRECAUTIONARY CONSUMPTION AND CYCLICAL CONSUMPTION FLUCTUATIONS

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SUMMARY

Theoretical consumption theory as Permanent Income Hypothesis (PIH) under the representative agent setting with permanent income innovation produces two consumption patterns that are not consistent with data observation. One is that the consumption growth rate is too volatile and the second is that the response of consumption is too insensitive to the lagged income change. Ludvigson and Michaelides (2001) attempted to use the buffer-stock saving model to solve the twin puzzles. Unfortunately, their simulated consumption series is still overly volatile and insensitive to the lagged income changes. In this dissertation, we investigate the buffer-stock saving model in detail to find out the reason of the failure of Ludvigson and Michaelides. We further improve the capability of buffer-stock saving model in resolving the consumption twin puzzles.

Consumption pattern is heavily affected by the perceived income process by households. In Chapter 1, we revisit the income process. Ludvigson and Michaelides (2001) presumed that the aggregate shock has a permanent effect on household income. However, by adopting LM test proposed by Lee and Strazicich (2003) and allowing for the presence of two break points in either drift or trend break, we do not detect a unit root in the aggregate income, which is consistent with the rejection to the panel unit root on PSID household real

log earnings data as studied in Pesaran (2007).

In Chapter 2, we first discuss the precautionary consumption behavior under complete and incomplete information structure by investigating the consumption policy function, the long-run stationary distribution and the impulse response function of expected consumption. We find that (1) precautionary consumption plus liquidity constraint will push gross wealth distribution skewed to the right; (2) precautionary consumption traces the pattern of income shock more closely in the complete information case; (3) with incomplete information, consumers will choose to suppress consumption further but this does not lead to a higher gross wealth level. Then, given the modified income process resulting from Chapter 1, we re-investigate the possibility of the buffer-stock model to resolve the consumption twin puzzles. Our results show that under complete information, the consumption-income relative smoothness ratio fits the data very well, but the model simulated consumption is still too insensitive to the lagged income. However, under incomplete information case, its smoothness ratio is lower, but the sensitivity coefficient becomes closer to data. The buffer-stock saving model does not fail in both dimensions as claimed by Ludvigson and Michaelides (2001).

In Chapter 3, we extend the research from the infinite life model in Chapter 1 and 2 to finite life span, and we also introduce altruism incentive across generations. We first compare the long-run features under various models. The observations are that in the finite life-span model, the marginal propensity of consume (MPC) becomes age-varying and higher than that in the infinite life model, which implies that short-run consumption fluctuation (volatility) will be higher than what we observe in the infinite life model. Then we re-do the calibration for the incomplete information case based on the finite life-span model and figure out that the finite life-span model indeed improves the results further, which is consistent with the long-run features.

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INTRODUCTION

In the consumption study, classical theory usually builds up a representative agent model to analyze aggregate consumption behavior. This model which embodies a quadratic utility function, infinite life horizon, stochastic labor income, and no restriction on borrowing and lending, produces a consumption pattern that satisfies Permanent Income Hypothesis (PIH).

PIH predicts that on the one hand, in response to a transitory income shock, the incentive to smooth their consumption stream over the life span implies that consumption from the current to the future will increase mildly and smoothly; on the other hand, in response to the permanent income shock, consumption will correspond in a one-for-one movement. Empirically, this implies that consumption growth would trace income growth closely as any income growth shock will lead to a permanent income level increase in the long run.

PIH also predicts that consumption will be orthogonal to the predictable or lagged income change. In other words, consumption change is forward-looking and should only be caused by the unpredictable income shock, which leaves consumption growth uninformative to the past income growth change. Any predictable income change will fully reflect upon the entire consumption stream plan.

However, there are two notable discrepancies between the model's predictions and aggregate data. One discrepancy is that, aggregate income data is observed to have a unit root and is usually modeled as first-difference-stationary AR(1) process with positive autocorrelation coefficient. According to the PIH model, consumption is predicted to be more volatile than income, because a positive income shock to its level signifies an even higher income level in the future, as a result, consumption will increase more than income to take advantage of the future rising income stream. But, in fact, econometrics studies show that aggregate consumption growth is much smoother than aggregate income growth.

The second discrepancy is that, consumption data is more sensitive to the lagged income change than the simulated consumption series from PIH. Ludvigson and Michaelides (2001) pointed out that the "correlation between consumption growth and lagged income growth is one of the most robust features of aggregate data".

In summary, aggregate consumption growth has been described as existing two puzzles: it is both "excessively smooth" relative to current labor-income growth, and "excessively sensitive" to lagged labor-income growth. So the challenge of consumption volatility study lies on reconciling the stylized facts from both micro and macro data observation.

There are several important progresses in consumption theory related to this topic after PIH inception. For example, Deaton (1991) built up a representative agent model of a liquidity constrained consumer and introduced four income process experiments to investigate the impact of income process on saving behavior and consumption volatility. His findings were that income process is a crucial element that affects the relative volatility of consumption to income and the more persistent the income shock is, the more volatile consumption will be. In particular, (1) when income is i.i.d, it is possible to smooth consumption with few assets as a buffer, the relative ratio of the standard deviation of consumption to income equals to 0.49, and "consumption is well predicted by income and starting assets, unrelated to lagged income"; (2) When income is level-stationary AR(1), assets are still used to buffer consumption, but "do so less effectively and at a greater cost in terms of foregone consumption". This is because given consumer impatience, "The smoothing of consumption over long autocorrelated swings requires more assets, and more sacrifice of consumption than is the case when income is i.i.d or negatively autocorrelated". The larger the positive autocorrelated coefficient of income stream, the less consumption is buffered[1]. The table 1 is excerpted from his paper, from which, we could observe that with the experimented AR(1) coefficient transferring from negative to positive, the relative volatility of consumption to income increases accordingly; (3) When income is a random walk, the agent just consumes his income with no asset left; (4) when income is

[1] In his paper, he named the motivation of decreasing consumption volatility as consumption buffered.

non-stationary, the growth rate mimics aggregate data and is positively serially correlated, saving becomes countercyclical.

[Table 1: Ratio of the standard deviation of consumption to the standard deviation of income for AR(1) income with different autocorrelation coefficient, abstracted from Deaton (1991)]

Deaton's attempt is suggestive in such a following way: firstly, he highlighted the fact that income process plays an important role in affecting consumption volatility; secondly, he tried to build a bridge between buffer saving and consumption volatility.

Our critique on Deaton's model is that his model missed household heterogeneity. Although his model built on a representative agent, however, only aggregate shock---but no idiosyncratic shocks---was introduced to this representative agent. In reality, microeconomic income processes are very different from their macroeconomic aggregates. Due to missing household heterogeneity, he failed to account for the main features of the aggregate time-series data. So we believe that it is necessary to bring the heterogeneous households into Deaton's model.

Carroll (1992, 1997) succeeded to build a buffer-stock saving model, which embodies idiosyncratic shocks and no liquidity constraint. In his model, he separated the idiosyncratic shock into two components: transitory and permanent. With this important step, he succeeded to explain how households use assets as a buffer to smooth the consumption against income shocks.

The natural question is that whether buffer-stock saving plus liquidity constraint could help to explain the twin “consumption puzzle” implied by PIH? Ludvigson and Michaelides (2001) tried to answer the question. They introduced both idiosyncratic and aggregate income shocks into the income process and imposed permanent shocks on both household and aggregate levels. Meanwhile, they assumed that households faced liquidity constraint. They compared their results under two scenarios: one was that households could observe each component of their earnings separately (complete information); the other was that households’ information set was incomplete, i.e. they could not distinguish aggregate from idiosyncratic shocks, but rather could only observe how much their income changed in a given period. Unfortunately, their simulated consumption series was still overly volatile and insensitive to the lagged income change.

Our critiques on Ludvigson and Michaelides (2001) are as follows: firstly, their income process setting is misleading. Although they introduced both aggregate and idiosyncratic shocks into the income process, however, the way that they imposed permanent shocks on both aggregate and household levels may be wrong; secondly, their paper tried to borrow the buffer-stock saving model to solve the “consumption excessiveness” puzzles but they did not explain why. In other words, the mechanism of how precautionary consumption behavior created by the buffer-stock saving model works to decrease consumption volatility is missing; thirdly, their conclusion that the buffer stock model is useless to solve the puzzles is debatable.

The contribution and novelty of this dissertation are as follows: in Chapter 1, we will revisit the income process; Chapter 2 will explain the mechanism of how precautionary consumption works to decrease the consumption volatility when households face uncertainty and then do the calibration to re-investigate the capability of the buffer-stock model in resolving the consumption twin puzzles; and Chapter 3 is the extension.

To see the thesis structure more clearly, we elaborate the relationships among such key words as the income process, precautionary consumption and cyclical consumption fluctuations as follows. Firstly, a stochastic income process will create uncertainty to households. In response to the uncertainty, households' consumption behavior will be adjusted to include precautionary attitude and liquidity constraints will enhance this kind of precautionary incentive. Secondly, this precautionary consumption behavior will help to decrease consumption volatility and therefore, have an impact on consumption fluctuations. Thirdly, how large the consumption fluctuations are depends on the income process setting. The more persistent the income shock we choose, the more volatile the calibrated consumption volatility we will observe. So to fit the actual data well, choosing correct income process setting becomes crucial.

CHAPTER 1

Income Process Re-investigation

1.1 Introduction

Deaton (1991) highlighted that the income process is a crucial element that affects the relative volatility of consumption to income. The more persistent the income shock is, the more volatile consumption will be. However, his model missed household heterogeneity. Only the aggregate shock--but no idiosyncratic shocks--was introduced to his representative agent model. Ludvigson and Michaelides (2001) improved on this part by introducing both idiosyncratic and aggregate income shocks into the income process but they imposed permanent shocks on both household and aggregate levels. Is it a correct way to do so? This is the question we want to answer in this chapter. Our conclusion is that placing the aggregate shock to the permanent component of household income seems to be implausible. Based on this observation, we reset the income process in which the aggregate shock is regarded as a transitory shock.

The rest of the chapter is organized as follows. Section 1.2 discusses two ways to decompose the income process; Section 1.3 does unit root tests on both household and aggregate levels; Section 1.4 investigates the correct way of income process setting based on unit root tests' results; Section 1.5 concludes.

1.2 Income process decomposition

There are two ways to decompose the income process. One way to decompose it is based on the question “who receives the shock?”. If the shock hits everyone uniformly, then this shock is defined as an aggregate shock. If a shock is household specific and no correlation across the households, then this shock is regarded as an idiosyncratic shock. For aggregate shocks, they normally show significant serial correlation in the high frequency (quarterly) data. The contribution of the aggregate shock to household income variation is relatively small compared to the idiosyncratic shock. Another way to decompose the income process is to check the persistence of the shock, by which, permanent shocks could be distinguished from transitory shocks. The permanent shock is a shock with unit root, for example, an unexpected job promotion. It is difficult to imagine that a job position can be promoted today but downgraded tomorrow. As a result, job promotion could be considered as a permanent shock to a personal life. On the contrast, transitory shock is a shock without unit root, for example, a temporary unemployment. Carroll (1992) estimated the standard deviation of the embodied permanent and transitory components without differentiating aggregate and idiosyncratic shocks. In Carroll’s estimation, for annual frequency data, the standard deviation for the shock to the income growth rate is about 10 to 12 percentage point (i.e. standard deviation of the permanent component) and to the log income level is about 15 to 17 percent (i.e. standard deviation of the transitory component).

The question is that, how to link these two decompositions with each other? i.e. what is the persistence of aggregate and idiosyncratic shocks? To answer this question, unit root tests on both aggregate and household levels are necessary.

1.3 Unit Root Tests

Proceeding empirical papers on household level panel data took a first-difference of the log income data prior to model estimation. The studies focused on the income growth rates. For example, MaCurdy (1982) estimated the income growth rates process using the panel data from *Panel Study of Income dynamics* (PSID). The model specification for the income growth rates is a moving average process. However, it would be dangerous to take the process from those micro studies as given for a macro scope analysis. With the first difference on data, the error term shocks would have a permanent effect on the income level unless further restrictions are imposed on the error term process. Consider a trend stationary log income (y) process as $y_t = a * t + \varepsilon_t$ with ε being i.i.d. Its first difference follows $\Delta y_t = a + \Delta\varepsilon_t$. Therefore, if we estimate a model $\Delta y_t = a + \mu_t$, the regression error μ_t must be restricted to follow a moving average process such that $\mu_t = \varepsilon_t - \varepsilon_{t-1}$. Researches such as MaCurdy generally estimated the model without the restriction $\varphi = 1$ for the error term process $\mu_t = \varepsilon_t - \varphi\varepsilon_{t-1}$. Adopting the unrestricted estimate result under finite sample is like imposing unit root in the household income process.

1.3.1 Unit root tests on household level

The unit root test on panel data (household level) has become popular since the late 90s[2]. Pesaran (2007) examined the PSID household real log earnings data. He considered the households with male heads aged 25-55 with at least 22 years of usable earnings data and separated these households based on their educational background into three subgroups: high-school dropouts, high-school graduates and college graduates. Under the panel unit root test that allowed for cross-section dependence, the unit root was rejected for the sample as a whole, but not for the subgroups of the high-school dropouts. The test results for the whole sample were consistent through various test statistics. The rejection to the panel unit root implies that not all household incomes are subject to a permanent shock. Therefore, placing the aggregate shock to the permanent component of household incomes seems to be implausible as adopted in Ludvigson and Michaelides (2001).

1.3.2 Unit root tests on aggregate level

To investigate whether aggregate income has a unit root or not, we

[2] Bowman (1999), Choi (2001), Hadri (2000), Im et al. (1995, 2003), Levin et al. (2002), Maddala and Wu (1999), and Shin and Snell (2002).

conducted the normal augmented DF test under a model with trend and drift. The data we collected is the real personal disposable income, net of dividend and interest incomes, from 1959:Q1 to 2008:Q4. The ADF test has a test statistics -2.97 which does not reject a unit root. This test result is inconsistent with the micro panel study where the presence of a unit root from the aggregate component is rejected. One possible reason is that structural break might have happened in the aggregate data series. Perron (1989) argued that many macroeconomic time series data if we observe a unit root may be only due to structural breaks they have, that is, one-time change in the level or slope of the trend function. If we remove these structural breaks, most time series are not characterized by the presence of a unit root. Fluctuations are indeed stationary around a deterministic trend function. Figures 1 and 2 show the time series plot of the log income series and the linear detrended log income series respectively. There seems to be an obvious trend breaking point in the early 70s (the trend line), which leads to a hump shape in the detrended income, that is, the deviation from the trend accumulated until early 1970s and then gradually declined. Furthermore, the 9-11 terrorist attack also had a strong impact on the data series, which caused a deviation jump-up in around 2001. However, since this deviation did not accumulate for a long time, it can be regarded as a level break.

[Figure 1: Log non-asset personal income]

[Figure 2: Linear detrended log non-asset personal income]

To allow for the presence of break points, we adopt another aggregate-level unit root test—the Lagrange multiplier unit root test proposed by Lee and Strazicich (2003). The reason why we choose this LM test from many alternatives that also allow for break points is that Lee and Strazicich's LM test allows for breaks under both the null and alternative hypotheses. Imagine if we assume no structural breaks under the null, such as Lumsdaine and Papell (1997), then a rejection of the null does not necessarily imply rejection of a unit root per se, but may imply a rejection of a unit root without break. Similarly, the alternative does not necessarily imply trend stationary with breaks, but may indicate a unit root with breaks. However, for the LS approach, since breaks are allowed under both the null and alternative hypotheses, a rejection of the null hypothesis unambiguously implies trend stationarity. The two break points are set at 1972:Q4 and 2001:Q1 in either drift or trend break. The LM test t-statistic is equal to -4.5068 which rejects a unit root under 5% significance level with exogenously specified break points.

We detrend the log income by regressing it with the constants and the time trends that include proper dummies and their interactions to reflect those two break points. The AR(1) autocorrelation coefficient estimate based on the detrended data is 0.853 with a standard deviation of 3.6 percent. The regression residual has a standard deviation of one percent.

In summary, the fact that the household level has a unit root does not necessarily mean that the aggregate level has a unit root. Conversely, if the aggregate level has a unit root then the household level must have a unit root, because this unit root is imposed on everyone. Based on both the macro and micro data studies, it seems to be reasonable that aggregate shocks are transitory. Some households possess a unit root in their income process, while some do not. Table 2 summarizes the relationship between these two decomposition methods.

[Table 2: Unit root test summary]

1.4 Income process setting

Based on the discussion above, income process can be specified as

$$Y_{i,t} = P_{i,t}(G_t V_{i,t}) \tag{1.1}$$

$$P_{i.t} = P_{i,t-1} N_{i,t} \tag{1.2}$$

where $V_{i,t}$ and $N_{i,t}$ are two types of idiosyncratic shocks, subscript *i* is introduced to denote the *i*th household; $\ln V_{i,t}$ and $\ln N_{i,t}$ are independently and identically distributed (i.i.d) normally distributed with mean zeros and variances σ_v^2 and σ_n^2 respectively. In particular, $V_{i,t}$ is a transitory shock and $N_{i,t}$ is a permanent shock, which introduces a permanent effect on income through the random walk process; Secondly, G_t is an aggregate shock, and since it is invariant across households, no household specific subscript *i* is needed; and based on the discussion above, G_t is regarded as a transitory shock and $\ln G_t$ follows an AR(1) process to fit U.S. quarterly data that have a significant serial correlation in quarterly frequency:

$$\ln G_t = \rho \ln G_{t-1} + u_t \qquad (1.3)$$

where AR(1) coefficient $0 < \rho < 1$ and u_t is i.i.d normal with mean zero and variance σ_u^2.

Taking logarithm on both sides, we get:

$$y_{i,t} = p_{i,t} + g_t + v_{i,t} \qquad (1.4)$$

$$p_{i,t} = p_{i,t-1} + n_t \qquad (1.5)$$

By taking the first difference, individual income growth follows:

$$\Delta y_{i,t} = n_{i,t} + \Delta g_t + \Delta v_{i,t} \qquad (1.6)$$

where the little case variable represents the log transformation of the capital letter variable and Δ represents the first difference. This first difference form is a little different from the one modeled in the previous literatures in which

$$\Delta y_{i,t} = n_{i,t} + g_t + \Delta v_{i,t}{}^{3} \qquad (1.7)$$

where both g and n components are permanent shocks, while the v component is a transitory shock. As a result, aggregate income will also possess

[3] Deaton (1991) and Ludivgson and michaelides (2001)

a unit root. However, in our setting, G_t is regarded as a transitory shock instead of a permanent one.

1.5 Conclusion

Ludvigson and Michaelides (2001) presumed that the aggregate shock has a permanent effect on household income. However, by adopting the Lagrange multiplier unit root test proposed by Lee and Strazicich (2003) and allowing for the presence of two break points in either drift or trend break, the hypothesis that aggregate income has a unit root is rejected, which is consistent with the rejection to the panel unit root on PSID household real log earnings data (Pesaran (2007)). Based on this observation, we reset the income process as $\Delta y_{i,t} = n_{i,t} + \Delta g_t + \Delta v_{i,t}$, where aggregate shock g is regarded as a transitory shock, the same as the idiosyncratic shock v. Another idiosyncratic shock n is set as a permanent shock.

CHAPTER 2

Precautionary Consumption and Cyclical Consumption Fluctuations

2.1 Introduction

In chapter 1, by adopting the Lagrange multiplier unit root test proposed by Lee and Strazicich (2003) and allowing for the presence of two break points in either drift or trend break, we rejected the hypothesis that aggregate income has a unit root. Based on this observation, we reset the income process and regarded the aggregate shock as a transitory shock. In chapter 2, based on the modified income process, firstly, we try to explain the mechanism of how precautionary consumption behavior created by the buffer-stock saving model works to decrease the consumption volatility, which is missing in Ludvigson and Michaelides (2001), though they tried to the borrow buffer-stock saving model to solve the "consumption excessiveness" puzzles. We find that by changing the expected consumption growth, precautionary consumption will decrease consumption volatility. Secondly, we re-do the calibration and find that buffer-stock model indeed fits the data better than PIH. In this chapter, we also discuss the consumption outcomes under two different income information structures: complete and incomplete information. We figure out that precautionary consumption traces the pattern of income shocks more closely in the complete than in the incomplete case. With incomplete information consumers will choose to suppress consumption further but this does not lead to a higher gross wealth level.

There are two main parts to the chapter. The first part focuses on discussing precautionary consumption behavior under complete and incomplete information structure. Subsection 2.2.1 discusses the relationship between precautionary consumption and uncertainty; Subsection 2.2.2-4 outlines the basic model; 2.2.5 introduces the information structure; 2.2.6 shows consumption policy functions under different information structures; 2.2.7 discusses how to measure the consumption suppression; 2.2.8-9 investigates the expected consumption growth and corresponding impulse response function. The second part moves to the calibration, Subsection 2.3.1 introduces the cyclical consumption fluctuations; 2.3.2 reports our model's calibration results and compares them with those in Ludvigson and Michaelides (2001) and in real data; Subsection 2.4 concludes.

2.2.1 Precautionary Consumption and Uncertainty

The direct impact of uncertainty on consumption reflects on expected consumption growth. This relationship can be shown from the Euler equation where the utility discount rate is a reciprocal of one plus the interest rate:

$$u'(C_t) = E_t u'(C_{t+1}) \tag{2.1}$$

After taking Taylor expansion on both sides:

$$u''(\bar{C})(C_t-\bar{C}) = E_t u''(\bar{C})(C_{t+1}-\bar{C}) + E_t \frac{u'''(\bar{C})}{2}(C_{t+1}-\bar{C})^2 \quad (2.2)$$

Reorganize it to:

$$\underbrace{(E_t C_{t+1}-C_t)/C_t}_{\text{Expected Consumption Growth Size}} = \underbrace{-E_t \frac{u'''(\bar{C})}{2u''(\bar{C})} \frac{(C_{t+1}-\bar{C})^2}{C_t}}_{\text{Uncertainty Size}} \quad (2.3)$$

The left-hand side of equation (2.3) demonstrates the expected consumption growth size and the right-hand side shows the uncertainty size. The second moment (uncertainty) will have an impact on the first moment (expected consumption growth), which leads to the consumption suppression.

[Figure 3: Expected consumption growth]

In Figure 3, the vertical line is the marginal utility of consumption and the horizontal line is the level of consumption. To make Euler equation hold, that is, to make marginal utility of consumption at time *t* equals to the expected marginal utility of consumption at time *t+1*, the consumption level at time *t* will be suppressed. Household must decrease his consumption and transform it into asset as a buffer. The gap between C_t and E_tC_{t+1} divided by C_t reflects the expected consumption growth and the consumption management that takes a precautionary measure against uncertainty.

The feedback of the second moment (uncertainty) to the first moment (expected consumption growth) is non-negligible only if the marginal utility curvature (i.e. $-u'''/u''$) is not trivial. So instead of using quadratic consumption utility function proposed by PIH, which cannot create convexity on its marginal utility curve, researchers usually propose CRRA form as $u(c) = c^{1-\chi}/(1-\chi)$, where χ is the CRRA coefficient and $\chi > 0$. The second necessary condition is that there is consumption uncertainty so that the conditional consumption variation $E_t\,(C_{t+1} - \bar{C})^2 > 0$. The second part usually results from income uncertainty and its magnitude is affected by the size of income uncertainty. As we know, the more persistent the income shock is, the more uncertain the income process will be. So the persistence of income shock plays an important role in determining the size of conditional consumption variation and therefore the size of expected consumption growth. This is why Deaton (1991) observed that when aggregate income is i.i.d, namely, no persistence of the shock, "it is possible to make consumption very much smoother than income without borrowing and without accumulating much assets". Due to lack of persistent income shock, even though his utility function is of a CRRA form, it still cannot create suppressing consumption, or say, asset does not increase. Furthermore, the magnitude of expected consumption growth is enhanced by the liquidity constraint, because if so, the fall in income will cause a large fall in consumption unless the individual has savings. Therefore, the presence of liquidity constraints causes individuals to save (suppress the consumption) against the effects of future falls in income.

2.2.2 Preference and Budget Constraint

Considering the following problem for each household:

$\text{Max } \sum_{k=0}^{\infty} E(\beta^k u(C_{i,t+k})|\Omega_{i,t})$

s.t.

$$X_{i,t+1} = R(X_{i,t} - C_{i,t}) + Y_{i,t+1} \quad (2.4)$$

$$Y_{i,t} = P_{i,t}(G_t V_{i,t}) \quad (2.5)$$

$$P_{i.t} = P_{i,t-1} N_{i,t} \quad (2.6)$$

$$X_{i,t} \geq C_{i,t} \quad (2.7)$$

For all t.

Household *i* at time *t* seeks to maximize his present discounted expected utility, where β is the discount factor; $\Omega_{i,t}$ is the information set that is up-to-date for the household. The contemporaneous utility function is assumed to be CRRA function so that $u(c) = c^{1-\chi}/(1-\chi)$ with $\chi > 0$.

Equation (2.4) is the budget constraint, which describes the evolution of the gross wealth. At the beginning of the period, the household holds $X_{i,t}$ unit of real wealth which includes the asset interest income and current income $Y_{i,t}$. Following Carroll (1997), we call *X* the gross wealth. With the gross wealth at time *t*, given $C_{i,t}$ unit of consumption, the end of the period wealth is $X_{i,t} - C_{i,t}$. Let R be asset return (interest rate plus one), the gross wealth in the next period is shown in equation (2.4).

Equation (2.5) and (2.6) are income process setting introduced in chapter 1.

In particular, $\ln V_{i,t}$ and $\ln N_{i,t}$ are i.i.d normally distributed with mean zeros and variances σ_v^2 and σ_n^2 respectively. For $\ln G_t$, it follows an AR(1) process

$$\ln G_t = \rho \ln G_{t-1} + u_t \qquad (2.8)$$

where AR(1) coefficient $0 < \rho < 1$ and u_t is i.i.d normal with mean zero and variance σ_u^2. The process suggests that the log income process $y_{i,t} = \ln Y_{i,t}$ can be expressed as $\Delta y_{i,t} = n_{i,t} + \Delta g_t + \Delta v_{i,t}$ where little case represents the log transformation of the capital letter variable and Δ represents the first difference. Therefore, $n_{i,t}$ is the permanent component, while g_t and $v_{i,t}$ are the transitory components.

Equation (2.7) is the liquidity constraint, which means that current consumption cannot exceed total current gross wealth. Note that consumption, however, can be less than or greater than $Y_{i,t}$, which corresponds to saving or dissaving respectively.

2.2.3 Parameter Setting

We begin by solving the model under a set of baseline parameter assumptions that fit the U.S. quarterly data:

[Table 3: Parameter setting]

We take $\rho = 0.85$ and $\sigma_u = 0.01$ based on the non-asset income series that is removed off time trend and drift. We allow for two break points present in the data. The serial correlation coefficient and the standard deviation imply that the standard deviation of aggregate component is 0.019. Based on the estimates of Carroll (1992) from *Panel Study of Income dynamics* (PSID), the standard deviation of permanent and transitory components are set to be 0.1 respectively. We follow the method as Ludvigon and Michaelides (2001) to convert the annual standard deviation into the quarterly standard deviation by dividing them by two (the variance by four), therefore, closely $\sigma_{\mathrm{n}} = 0.05$ and $\sigma_{\mathrm{v}} = 0.05$. These parameters are also close to the parameter values in Ludvigon and Michaelides (2001). Besides the income process parameters, we assume that consumers are impatient, $\beta R < 1$. This assumption is necessary because consumers for whom $\beta R \geq 1$ will accumulate assets indefinitely which does not appear to be the case for many consumers, just as mentioned in Deaton (1991). "[When $\beta R \geq 1$], in the limit, the income stream becomes irrelevant as consumption comes to be financed increasingly out of capital income". Furthermore, if the agents are patient, borrowing constraint is irrelevant any more, because "saving, not borrowing, is their main concern". We choose $\beta = 0.99$ and asset return (1+interest rate) R=1, i.e. the baseline interest rate is set to be zero following Carroll (1992). The CRRA coefficient is set to $\chi = 2$ as usual. Each unit of time represents a quarter.

2.2.4 Euler Equation

Following Carroll (1997), we detrend the gross wealth accumulation process by dividing it by the permanent component. Let $x_{i,t} = X_{i,t}/P_{i,t}$ and $c_{i,t} = C_{i,t}/P_{i,t}$. Then

$$x_{i,t+1} = R(x_{i,t} - c_{i,t})/N_{i,t+1} + G_{t+1}V_{i,t+1} \tag{2.9}$$

As Deaton and Guy Laroque (1992), we assume that $\beta RE_t(N_{i,t+1}^{-\chi})<1$ holds[4], which guarantees a unique solution to consumption policy function. The consumption policy function will satisfy

$$u'(C_{i,t}) = \max\{u'(X_{i,t}), \beta RE_t u'(C_{i,t+1})\} \tag{2.10}$$

Given the CRRA utility function, the above first order condition can define on the detrended variables such that

$$u'(c_{i,t}) = \max\{u'(x_{i,t}), \beta RE_t u'(c_{i,t+1}N_{i,t+1})\} \tag{2.11}$$

That is, if liquidity constraint is not binding, household will choose to consume at such a level that marginal utility of consumption at time *t* equals to the effectively discounted expected marginal utility of consumption at time *t+1*. However, if liquidity constraint is binding, then household can only consume his current gross wealth.

2.2.5 Information Structure

We assume that households can distinguish transitory shocks from

[4] Given that $\ln N_{i,t}$ is i.i.d normal with mean zero and variance σ_n^2, this condition is equivalent to $\beta R\exp\left\{\frac{\chi^2\sigma_n^2}{2}\right\} < 1$

permanent shocks. For transitory shocks, we consider two situations based on whether or not the household can distinguish the aggregate shock G_t from the idiosyncratic shock $V_{i,t}$. If the household can distinguish each component, then the household has complete information. If not, the household has only incomplete information and observes $(GV)_{i,t} = G_t V_{i,t}$ as a whole in each period.

For incomplete information, at time *t*, the household makes an inference of expected $\log(G_{t+1}V_{i,t+1})$ based on $\log(G_t V_{i,t})$. This implies that household adopts a learning process by projecting $\log(G_{t+1}V_{i,t+1})$ on $\log(G_t V_{i,t})$, which under normality assumption can be expressed as an AR(1) learning process as

$$\log(G_{t+1}V_{i,t+1}) = \varphi \log(G_t V_{i,t}) + \varepsilon_{i,t+1} \tag{2.12}$$

where $\varphi = \rho \frac{\sigma_g^2}{\sigma_g^2 + \sigma_v^2}$, $\varepsilon \sim \text{iid } N(0, (1-\varphi^2)(\sigma_g^2 + \sigma_v^2))$ and σ_g^2 is the unconditional variance of $\ln G$.

Given $\log G_t = \rho \log G_{t-1} + u_t$ with u iid $N(0, \sigma_u^2)$, the unconditional variance of $\log G$ is

$$\sigma_g^2 = \frac{\sigma_u^2}{1-\rho^2} \tag{2.13}$$

When information is complete, $\log(G_{t+1}V_{i,t+1}) = \rho \log G_t + \eta_{i,t+1}$, where $\eta_{i,t+1} = \log(V_{i,t+1}) + \varepsilon_{i,t+1}$. For a one percent increase in the aggregate shock, household will expect future transitory component to increase by ρ percent. For the incomplete information case, household will expect future transitory component to increase by φ percent. Given the parameters set above, we could calculate this perceived transitory shock persistency $\varphi = 0.107$ much less than the aggregate shock persistency $\rho = 0.85$. Therefore, we would expect incomplete information to decrease the effect of the aggregate shock on consumption. However, the uncertainty under incomplete information is $(1-\varphi^2)(\sigma_g^2+\sigma_v^2)$ which is larger than that under complete information (i.e. $\sigma_u^2+\sigma_v^2$) [5]. The household would have a stronger incentive to build up precautionary saving level.

Under incomplete information in response to a transitory shock driven by the aggregate force, the household will think that the income change has a very short life. Consumption smoothing motivation will not trigger much consumption response. As a result, aggregate consumption is expected to be smooth. Under PIH, the volatility ratio is close to zero. Things are a little different when households are impatient and prudent. We will discuss it in the following section.

2.2.6 Consumption Policy Function

The dynamic programming problem for our consumption model does not have a known explicit function solution form, so we solve the model numerically. The numerical procedure is recursive on a discretized space. For underlying normal distributed shocks, their supports are discretized into 11

[5] With proper rearrangement, it is true that $(1-\varphi^2)\left(\sigma_g^2+\sigma_v^2\right)-(\sigma_u^2+\sigma_v^2) = \rho\sigma_g^2\left(1-\frac{\sigma_g^2}{\sigma_g^2+\sigma_v^2}\right) > 0$

grids evenly that covers a three-standard-deviation range. The aggregate AR(1) shock process is also described by a ten-point discrete Markov process. The consumption policy is a function of the detrended gross wealth x and a shock state variable which is G if the information is complete or (GV) if the information is incomplete. We use S to denote this income innovation state. For the gross wealth space, it is descretized over [0.01 2] with 50 even grids. As in Deaton (1991), the recursion can be thought of as the backward solution to a finite life stochastic dynamic program. The initial policy function is set to $C(x, S) = x$. Thereafter, by using backward-recursive substitution method, we use (2.11) subject to (2.9) to recursively update the function until it converges. The convergence criterion is set to ensure that the updating gain on each grid is no larger than 0.01 percent. The details of the method of numerical solution are contained in Appendix 1 & 2.

[Figure 4: Consumption policy functions of complete/incomplete information]

Figure 4 are the consumption policy functions drawn by ourselves magnified from original policy functions created by Matlab. The reason why we do not use Matlab graphs themselves is that the function features shown in the original graphs are not clear enough to see. For analytical convenience, we magnify and draw the functions by ourselves. $\bar{s}$ represents the mean level of lnG for complete information and of ln(GV) for incomplete information. Δs indicates the 1 percent standard deviation from the mean level. The first feature we want to highlight is that the sensitivity of consumption to the shock decreases with X increases. That is, given the same unit of positive shock, the amount of consumption increase is decreasing with X increase. From the figure, we could observe that the consumption change becomes narrow when X increases. The second feature worth noticing is that node "A" where consumption policy function branches out from the 45 degree line represents the maximal gross wealth level that the household is liquidity constrained. When the household is experiencing a sequence of bad shocks, its gross wealth level will fall. However, "A" threshold falls as well. As a result, it mitigates the chance of getting liquidity constraint. This reflects the precautionary incentive that keeps the household from experiencing more volatile consumption once it is liquidity constrained. The third thing worth highlighting is from the comparison of policy functions between complete and incomplete information. In the figure, blue line represents that consumption adjusts when shock G changes by one unit in complete information case, correspondingly, red line indicates the new level of consumption as shock GV changes by the same unit in incomplete information case. From the figure, it is obvious that consumption adjustment is smaller (the gap between two red lines is narrower) in the incomplete information than in the complete information case. This is because that the shock persistency is perceived to be smaller in the incomplete information case.

2.2.7 Consumption Suppression

To show the consumption suppression due to uncertainty, we consider its consumption level relative to the gross wealth in the long run. By long run, we mean their unconditional expected levels under long-run stationary distribution. For consumption, its long-run level $\mathrm{Ec(x,s)} = \int_{\mathrm{x,s}} \mathrm{c(x,s)f(x,s)dxds}$; for gross wealth, we consider $\mathrm{Ex} = \int_{\mathrm{x,s}} \mathrm{xf(x,s)\,dxds}$ where f(x, s) is pdf.

Given the consumption policy function, we can derive the stationary joint distribution of gross wealth and the state variable for consumption. Stationary distribution, simply speaking, is the distribution at which the economy will gradually settle down. A stationary distribution $\mathrm{F(x', s')} = \mathrm{Pr}(\mathrm{X}' < \mathrm{x}', S' < s')$ with its corresponding PDF $\mathrm{f(x', s')}$ satisfies

$$\mathrm{F(x', s')} = \int_{\mathrm{x,s}} \mathrm{Pr}(\mathrm{X}' < \mathrm{x}', \mathrm{S}' < \mathrm{s}' | \mathrm{X} = \mathrm{x}, \mathrm{S} = \mathrm{s})\mathrm{f(x,s)dxds} \tag{2.14}$$

for all possible (x’,s’) in the support; where $\mathrm{X}' = \frac{\mathrm{R}(\mathrm{X}-\mathrm{c}(\mathrm{X,S}))}{\mathrm{N}'} + \mathrm{G}'\mathrm{V}'$ is defined by the gross wealth accumulation equation. The *prime* symbol represents the next period. X is the detrended gross wealth level (which is in little case before). Capital letter represents a random variable and its little case represents a specific outcome. Consumption is a function of both gross wealth

X and a shock state variable S which is G for complete information and (GV) for incomplete information.

We solve for the stationary distribution numerically. First, we inherit the discretized spaces for N, G and V we used before for solving the consumption policy function. Secondly, we discretize the gross wealth space X from 0.5 to 2.5 into 100 grids. The choice of range is wide enough so that the probability of reaching the boundary is zero in our computation. The detail of the computation is stated in the Appendix 6 and 7. Figure 5 shows stationary joint distributions over $X \times S$ space under complete/incomplete information cases respectively, which indicate given specific S and X, what the corresponding probability is.

[Figure 5: Stationary joint distributions under complete/incomplete information cases]

[Figure 6: Contour plots of joint probability under complete/incomplete information cases]

Figure 6 demonstrates contour plots of joint probability over $X \times S$ space under complete/incomplete information cases respectively. The shape of each contour line stretches from the lower left towards the upper right. This shows the positive correlation between income innovation and the gross wealth. The correlation is stronger in the incomplete information case. Furthermore, the contour lines are denser for low gross wealth levels. This means that the correlation between income innovation and the gross wealth is stronger for the low wealth group. This is because liquidity constraints put a sharp drop of probability on the left.

[Figure 7: Stationary distributions of gross wealth under complete/incomplete information cases]

Figure 7 shows the stationary distributions of gross wealth under complete/incomplete information cases. The interesting thing is that though both aggregate and idiosyncratic shocks are symmetric, the distribution is skewed to the right for both information structures. The reason behind is that since the sensitivity of consumption to income shock declines with X increase as we see in Figure 4, with the same unit of income shock, consumption adjustment will become smaller, which will lead to a even higher X level at time $t+1$. As a result, extreme gross wealth is more likely to occur, which makes the distribution skewed to the right. Comparing both cases, in the incomplete information case, this is truer and leads more households to have their wealth levels further away from the median.

[Table 4: Long-run mean level and standard deviation of gross wealth and consumption in complete/incomplete information cases]

Given the consumption policy function and stationary joint distribution, we could calculate the long-run mean level and standard deviation of detrended gross wealth and consumption to investigate the consumption suppression. Table 4 shows the results. As we can see, the gap between the gross wealth and consumption is the precautionary saving level. In particular, with complete information, the mean gross wealth level is 3 percentage points higher than with incomplete information. Its standard deviation is larger as well, by about 7 percentage points. For consumption, the mean consumption level is higher under complete information than under incomplete information by a significant 20 percentage points. The significance of the mean consumption shows that even though the aggregate income shock does not account for a larger portion of household income variation, it is actually important for precautionary consumption management. This is different from the proceeding argument from Pischke (1995) where Pischke argued that due to its small variance contribution, aggregate shock information is negligible and households have no incentive to distinguish aggregate shocks from the idiosyncratic ones. But as we can see here, this extra information improves consumption strategy that leads to a higher mean level in the long run. We can also observe from the table that the consumption standard deviation is smaller under complete information case, that means, with better information, consumption fluctuations can be better buffered.

2.2.8 **Expected Consumption Growth**

In the PIH model, the consumption growth rate will change with unpredictable income changes. However, as for the expected consumption growth rate, it is predicted to be shock irrelevant. This is because according to the Euler equation,

$$E_t \Delta \ln C_{t+1} = \frac{1}{\chi} \ln R\beta \approx 0 \tag{2.15}$$

That is, the expected consumption growth rate is a constant in the PIH model and since R and β are set to be 1 and 0.99 respectively, $E_t \Delta \ln C_{t+1}$ is close to zero. However, the expected consumption growth rate under precautionary consumption management is time-varying. Figure 8 is the corresponding figure from one simulated random sample.

[Figure 8: Expected consumption growth rate under precautionary consumption management]

Parker and Preston (2005) used the consumption Euler equation to derive a decomposition of consumption growth into 4 sources: unpredictable information change, the intertemporal substitution effect (R), utility shifters and changes in consumption risk (precautionary saving + liquidity constraint). In our model, R=1 is fixed over time and so is the CRRA utility form as well, then the precautionary incentive is the only factor that has an impact on the expected consumption changes. In our calibration, 22% of actual consumption growth variation is expected. This is contrary to 0% contribution of expected consumption changes to total variation under PIH.

2.2.9 Impulse response of the expected consumption

Under the PIH model, in response to an income innovation, the consumption stream will shift upwards to match the extra permanent income that is generated. As a result, the impulse response of expected consumption $E(C_{t+k}|\Omega_t)$ for the shock happened at time *t* is a one-time level shift, leaving a flat impulse response shape. This is not true where there is a precautionary consumption.

The construction of the impulse response of expected consumption is as follows. Given the policy function $c(x, s)$ where x is the gross wealth and s is the income innovation state (which is G for complete information and GV for incomplete information), the computation of $E_0 c(x_t, s_t)$ is defined as

$$E_0 c(x_1, s_1) = \int_{x_1, s_1} c(x_1, s_1) \Pr(x_1, s_1 | x_0, s_0) \tag{2.16}$$

Recall that $x_1 = \frac{R(x_0 - c_0(x,s))}{N_1} + G_1 V_1$, we denote a function $W(x, s) = R(x - C(x, s))$. Then $x_1 = \frac{W(x_0, s_0)}{N_1} + G_1 V_1$ which shows that gross wealth equation is simply a mapping from $(x_0,\ s_0,\ G_1 V_1,\ N_1)$ to R^+. In other words, we can write $x' = f(x_0, s_0, G_1 V_1, N_1)$. Therefore, the expectation that we want to compute is

$$E_0 C(x_1, s_1) = \int_{N_1, G_1 V_1} C(f(x_0, s_0, N_1, G_1 V_1), G_1 V_1) \Pr(N_1, G_1 V_1 | x_0, s_0) =$$

$$\int_{N_1, G_1 V_1} C\left(N_1, G_1 V_1 | x_{0,} S_0\right) \Pr(N_1, G_1 V_1 | x_0, s_0) \tag{2.17}$$

The one period forward conditional expectation of which is a function of (x_0, s_0), i.e. $E_0 C_1 = \varepsilon(x_0, s_0)$. Therefore, $E_k C_{k+1} = \varepsilon(x_k, s_k)$. Applying Law of Iterative Expectation, we know that

$$E_0 C_{k+1} = E_0 (E_1 E_2 \dots E_{k-1}) \varepsilon(x_k, s_k) \tag{2.18}$$

As we can see that the key to compute the expected consumption impulse response lies on the computation of layers of expectation operator $E_1 E_2 \dots E_{k-1}$ on a function that is defined on (x,s) space. And each layer of computation is

$$E_k C(x_{k+1}, S_{k+1}) = \int_{x_{k+1}, S_{k+1}} \varepsilon(x_{k+1}, s_{k+1}) \Pr(x_{k+1}, s_{k+1} | x_k, s_k) \tag{2.19}$$

[Figure 9: Impulse response to a 1% positive GV shock under complete and incomplete information cases]

Figure 9 shows the impulse response to a 1% positive G shock under complete and incomplete information cases. Instead of jumping once and keeping the level flat under PIH, the expected consumption traces the income process very closely. In the complete information case, it jumps 0.95% immediately and gradually declines, which is consistent with what we observe from reality that when consumers face a positive transitory shock, they normally increase the consumption expenditure more in the short horizon than in the long horizon. And because of that, their expected consumption growth rate becomes negative, which is also observed from the figure. Meanwhile, to compare the impulse response functions under different scenarios, expected consumption will deviate more under complete information (0.95%) than under incomplete information case (0.25%) and trace income process more closely in the former case. This is because households perceive the shock to be more persistent in the complete information case.

[Figure 10: Impulse response to a 1% negative GV shock under complete and incomplete information cases]

The impulse responses of expected consumption are not symmetric. When a household faces a 1% negative shock, he will decrease his consumption level by 0.65% immediately and then revert to the mean level in complete information case (see Figure 10). As we can see, the decline of the consumption level is not as much as income declines, this is because he has a precautionary saving as a buffer. But since the household knows clearly about the persistence of the shock, he has to adjust the consumption level to prevent him from hitting the liquidity constraint. The interesting thing happened in the incomplete information case is that unlike complete information case where the household deviates his consumption downside from the mean level immediately, he is more reluctant to decrease his consumption at the beginning. With time passing, consumption gradually decreases and then reverts to the mean level, leaving a U-shape impulse response curve. This is also because at the onset, households think that the transitory shock driven by the aggregate force has a very short life, the perceived AR(1) coefficient $\varphi = 0.107$ which is much less than the actual one $\rho = 0.85$, but he gradually learns the actual persistency of the shock from the passing time and decides to decrease his consumption to avoid hitting the liquidity constraint.

2.3.1 Cyclical Consumption Fluctuations

Consumption theories such as the Permanent Income Hypothesis whereby the aggregate income has a unit root produce two consumption patterns that are not consistent with data observations. One pattern is that the consumption growth rate is too volatile and the other is that the response of consumption is too insensitive to the lagged income change.

Table 5 shows relative smoothness and excess sensitivity values computed from U.S. aggregate data. There are two major differences between our figures and Ludvigson and Michaelides (2001)'s figures. First, the sample periods are different. Their data range from 1947 to 1999. Ours range from 1959 to 2008; Secondly, we use non-asset income instead of labor income, as in the model the income source is not specific and, hence, shall not be limited to labor income. Following Ludvigson and Michaelides (2001), the smoothness ratio in the table is defined as the ratio of the standard deviation of aggregate consumption growth to the standard deviation of aggregate income growth. Excess sensitivity is defined as OLS coefficient of aggregate consumption growth on lagged aggregate income growth. We choose three kinds of consumption expenditure growth to calculate. $\Delta C_t/C_{t-1}$ represents the growth in nondurables and services expenditure. $\Delta C_t^{ND}/C_{t-1}^{ND}$ indicates only nondurable expenditure growth. $\Delta C_t^{S}/C_{t-1}^{S}$ is service expenditure growth. U.S. real data show that smoothness ratios are around 0.7 for total (nondurables + service) and nondurables expenditure growth and 0.46 for service expenditure growth. All figures are significantly less than 1. Consumption is actually less volatile than income. Meanwhile, sensitivity coefficients are around 0.21 for total consumption growth, 0.18 for nondurable goods consumption and 0.09 for service. All the numbers are significantly larger than 0. Consumption is moderately sensitive to the lagged income change. However, for PIH with a unit root in the aggregate income process, the simulated results are that for the smoothness ratio, it is 1.26, which means the simulated consumption is much more volatile than the simulated income; and for sensitivity coefficient, it is 0.00---simulated consumption is orthogonal to the lagged income change.

[Table 5: Relative smoothness and excess sensitivity: U.S. aggregate quarterly data (1959:Q1—2008:Q4)]

To solve the puzzles, Ludvigson and Michaelides (2001) borrowed the buffer-stock saving model to investigate whether this model could help to overcome the twin divergences. In their paper, they set the aggregate shock as a first-difference AR(1) process with ρ=0.23.

[Table 6: Relative smoothness and excess sensitivity: Ludvigson and Michaelides (2001) simulated results]

Table 6 shows their calibration results. They found that under complete information, for relative smoothness, its smoothness ratio is 1.09, still above one but lower than 1.26 from PIH. Meanwhile, its sensitivity coefficient, 0.055, is too insensitive compared to the real data and is similar to PIH result. In the incomplete information, however, the smoothness ratio declines to 0.91, but is still too volatile and the coefficient jumps upwards to 0.433, which becomes too sensitive. So they concluded that though incomplete information case produces aggregate consumption growth that is less volatile than the benchmark PIH and generates excess sensitivity to expected income growth, it still falls short of matching the data. Consumption growth is both too volatile and too highly correlated with lagged income growth. As a result, the buffer-saving model still has problems at explaining these two puzzles.

What we want to argue is that the unsatisfactory result may not be due to the failure of the precautionary consumption model, instead, in the above, we provide the evidence that precautionary consumption could help to decrease consumption volatility. So the failure may be because their income process setting is wrong. Since the aggregate income shock is set to be a permanent shock in their model, which leads the income to increase over time to a permanently higher level. Therefore, at the first time when a shock happens, the households know that this is just the beginning of an income-increase phase. They will increase consumption more than the current income increase to reflect future income raise. However, in our model, aggregate income shock is transitory and set as a detrended AR(1) process with $\rho=0.85$. As a result, the relative smoothness ratio is expected to be lower than one.

2.3.2 Calibration

For calibration, we use the same parameters set as shown above and simulate 100 samples, each of which consists of a panel of 2000 households across 150 periods. The number of households is determined by increasing the population until such a point that individual income draw once aggregated will match U.S. actual aggregate income process. This procedure shows that 2000 households are sufficient; using more households does not change the results. The first 50 periods are dropped to eliminate the impact of initial wealth condition. Calibration details refer to Appendix 4. Table 7 demonstrates our model's simulated results.

[Table 7: Relative smoothness and excess sensitivity: our model's simulated results]

In our model results, under complete information, the smoothness ratio is 0.64, which fits the data very well, but consumption is still too insensitive to the lagged income. On the other hand, under incomplete information case, its smoothness ratio is lower, but sensitivity coefficient becomes closer to data. The reason behind is that incomplete information causes a misperception of the persistence of the shock in our model. The perceived persistency $\varphi = 0.107$ is less than actual persistency $\rho = 0.85$. This misperception will lead to the result that households do not raise consumption by the amount that is warranted by the actual persistency of the shock, and therefore lower the relative consumption volatility to income. Meanwhile, the misperception will also create a sluggish response of consumption to the aggregate income shocks; the next period's consumption will be raised again when income is higher than expected. The sluggishness of consumption in turn produces a direct correlation between consumption growth and lagged income growth.

2.4 Conclusion

Precautionary consumption will decrease the consumption volatility by changing the expected consumption behavior. In particular, based on the simulation results, we find that (1) precautionary consumption plus liquidity constraint will enhance gross wealth distribution skewed to the right; (2) precautionary consumption traces the pattern of income shock more closely in the complete information case; (3) with incomplete information, consumers will choose to suppress consumption further but this does not lead to a higher gross wealth level. For the calibration, under modified income process, we find out that the buffer-stock model fits the data better than the PIH model. In particular, under complete information, the consumption-income relative smoothness ratio fits the data very well, but the model simulated consumption is still too insensitive to the lagged income. However, under incomplete information case, its smoothness ratio is lower, but sensitivity coefficient becomes closer to data. The buffer-stock saving model does not fail in both dimensions as claimed by Ludvigson and Michaelides (2001).

CHAPTER 3

Extension: Life Span, Altruism and Consumption Fluctuations

3.1 Introduction

From Chapter 1 to 2, we investigate the proper modeling of the income process, explain the mechanism of how precautionary consumption works to decrease consumption volatility and calibrate the smoothness ratio and sensitivity coefficient based on the modified income process. The calibration results become more consistent with the data. However, all the results are based on an infinite horizon model following Deaton (1991) and Ludvigson and Michaelides (2001), which is far from the reality. The adoption of an infinite life span is mostly for technical convenience to derive relatively simple stationary policy rules, but some characteristics of the finite life plan are missing. To be closer to the reality, this chapter extends the model from an infinite horizon to a finite life span and introduces altruism incentive, which is a bridge between the life span and infinite horizon model. Another purpose of doing extensions comes from the attempt of improving the calibration results under the incomplete information case discussed in Chapter 2, where its corresponding simulated smoothness ratio is 0.341, lower than those in complete information case and in real data but its sensitivity coefficient improves to 0.1113, much closer to reality. In this chapter, we will re-do the calibration for the incomplete information case in a finite life-span model to investigate whether the new model will help to improve the results further.

The first part of this chapter concentrates on comparing the long-run features under various models, such as the infinite life model and the finite life model with/without altruism concern. The discussion of these features will provide in-depth insight to the mechanism of how the finite life-span model and the altruism incentive work to adjust the sensitivity of consumption to income. In the second part, we will investigate whether these long-run features will have an influence on calibration results.

The rest of the paper is organized as follows. Section 2 sets up the finite life-span model and introduces altruism incentives; Section 3 introduces the marginal income propensity to consume and explores the life-cycle consumption patterns under various cases; Section 4 calibrates consumption fluctuations; Section 5 concludes.

3.2 The finite life-span model

3.2.1 Basic model structure

The main feature of the finite life-span model is to introduce 'Age' as a new dimension into the basic infinite model. Different from the infinite horizon model, in which households live forever, finite model assumes that a household's life is limited, which is closer to the reality, and on account of that, consumption behavior will be affected under this new setting.

We re-consider the following problem for a representative agent.

$$\text{Max} \sum_{t=0}^{T-A} E_0(\beta^t u(C_{i,t,A})|\Omega_{i,t,A})$$

s.t.

$$X_{i,t+1,A+1} = R(X_{i,t,A} - C_{i,t,A}) + Y_{i,t+1,A+1} \tag{3.1}$$

$$Y_{i,t,A} = P_{i,t,A}(G_t V_{i,t}) \tag{3.2}$$

$$P_{i,t,A} = P_{i,t-1,A} N_{i,t} \tag{3.3}$$

$$C_{i,t,A} \leq X_{i,t,A} \tag{3.4}$$

where subscript *A*, *i* and *t* denote age, the *i-th* household and time respectively. T is the maximal age of the household.

Household *i* with age *A* at time *t* seeks to maximize his present discounted expected utility, where β is the discount factor, $\Omega_{i,t,A}$ is the information set that is up-to-date for the household. The contemporaneous utility function is also assumed to be CRRA function so that $u(c) = c^{1-\chi}/(1-\chi)$ with $\chi > 0$.

Similar as described in Chapter 2, equation (3.1) is the budget constraint, which shows the evolution of the gross wealth. At time *t*, the *i*th household at age *A* holds $X_{i,t,A}$ unit of real wealth which includes the asset interest income and current income $Y_{i,t,A}$. Following Carroll (1997), we call *X* the gross wealth. With the gross wealth at time *t*, given $C_{i,t,A}$ unit of consumption, the end of the period wealth is $X_{i,t,A} - C_{i,t,A}$. Let R be asset return (interest rate plus one), the gross wealth in the next period is shown in equation (3.1).

Equation (3.2) and (3.3) are income process setting. $\ln V_{i,t}$ and $\ln N_{i,t}$ are i.i.d normally distributed with mean zeros and variances σ_v^2 and σ_n^2 respectively. In particular, $V_{i,t}$ is a transitory shock, and $N_{i,t}$ is a permanent shock which introduces a permanent effect on income through the random walk process of $P_{i,t,A}$. The difference of income process settings between here and in chapter 1 and 2 is that, subscript *A* is introduced to labor income $Y_{i,t,A}$ and the permanent component $P_{i,t,A}$. This allows for the flexibility of incorporating the age specific component in the labor income process. Since G_t is an aggregate shock and invariant across households, no household specific subscript *i* and age subscript *A* are needed. Similar to the previous chapters, G_t is regarded as a transitory shock.

One feature of income process under finite horizon setting is the link between the oldest of current generation and the youngest of the next generation. Since the next generation is the continuity to the current one in the finite life-span model, we treat the income of the youngest of the next generation Y_1^{next} as Y_{T+1}. It means that $Y_{1,t}^{next} = P_{1,t}^{next} V_{1,t}^{next}$ where $P_{1,t}^{next} = P_{T,t-1} N_{1,t}^{next}$ and $P_{T,t-1}$ is from the current generation within the same household.

Equation (3.4) is the liquidity constraint. The consumption of the i^{th} household with age *A* at time *t* cannot exceed this household's gross wealth. Note that $C_{i,t,A}$ can be less than or greater than $Y_{i,t,A}$, which corresponds to saving or dissaving respectively.

3.2.2 Parameter setting

To analyze the long-run feature, we begin by solving the model under a set of baseline parameter assumptions that fit the U.S. annual data:

[Table 8: Parameter setting fit with annual frequency]

In calibration for the annual data frequency, G_t is i.i.d log normal as well—no serially correlated shock. This implies that whether the household can distinguish G and V does not matter. Here, we assume imperfect information. For annual frequency, we take $\sigma_n = 0.1$, $\sigma_v = 0.0968$ and $\sigma_g = 0.025$. These parameters setting implies that the standard deviation of $\log(GV)$ is 0.1 which is the same as the setting in Carroll (1997) where there is no aggregate shock G but log(V) has a standard deviation of 0.1. β is time discounting factor, set to be 0.96 for annual frequency. χ is the risk aversion parameter, equal to 2 as usual; T is the maximal age of household in the society where we choose T=70.

3.2.3 Euler equation

Following Carroll (1997), we detrend the gross wealth accumulation process by dividing it by the permanent component. Let $x_{i,t} = X_{i,t}/P_{i,t}$ and $c_{i,t} = C_{i,t}/P_{i,t}$. Then

$$x_{i,t+1,A+1} = R\left(x_{i,t,A} - c_{i,t,A}\right)/N_{i,t+1} + G_{t+1}V_{i,t+1} \tag{3.5}$$

The consumption policy function will satisfy

$$u'\left(C_{i,t}\right) = \max\{u'\left(X_{i,t,A}\right), \beta R E_t u'(C_{i,t+1,A+1})\} \tag{3.6}$$

Given the CRRA utility function, the above first order condition can be defined on the detrended variables such that

$$u'\left(c_{i,t,A}\right) = \max\{u'\left(x_{i,t,A}\right), \beta R E_t u'(c_{i,t+1,A+1}N_{i,t+1})\} \tag{3.7}$$

As in Deaton (1991), the recursion can be thought of as the backward solution to a finite life stochastic dynamic program. The initial policy function is set to $c_T = x_T$. Thereafter, by using backward-recursive substitution method, we use (3.7) subject to (3.5) to recursively compute the optimal consumption at each age. The details of the method of numerical solution are contained in Appendix 3.

3.2.4 Altruism attitude

However, from the infinite life model to the finite life span model without altruism concern, it seems to go from one extreme case to the other. The former case assumes that households could live forever, which is unrealistic but the latter one is also far from reality because although it assumes that a household has a finite life, but he does not concern anything about his following generation. As a result, the consumption response towards the later age is likely to be more overstated than the reality truth. To resolve this issue, we introduce altruism attitude into the finite life span model.

We assume one household that has a finite lifetime (say the maximal age is 70). By the day he/she dies, each person gives birth to a new life. When the agent is selfish, his utility does not take into account of the utility of the future generation. As a result, the consumption programming problem is simply a finite horizon problem. However, when the agent cares about the future generation, his utility will be determined not only by his consumption level but also by the value of the future generation. We allow the degree of altruism to vary in the way that the value of the household depends on both his utility of consumption and the value of the new born.

The modeling of altruism can be described by the following Bellman equation faced by the oldest person in each generation:

$$V_T(X_T) = u(C_T) + \alpha\beta E_T V_1^{next}(X_1^{next}) \tag{3.8}$$

where T means the last period of this generation, superscript 'next' and subscript '1' together mean the first period of the next generation and α is the degree of altruism, between 0 and 1. In the extreme case, $\alpha = 0$ which means complete selfish, in which household will run out of his wealth when he dies; $\alpha = 1$ represents complete altruism, i.e. he will consume the same unit as usual and leave all the left to the next generation. In reality, α is most likely to lie between 0 and 1, meaning partial altruism, that is, he may consume more than usual but still leave the surplus as a bequest to his child. The larger α is, the higher degree of altruism is. The first order condition (FOC) of the consumption at the last living day of an agent will be:

$$\mathrm{F.O.C.}\quad u'(C_T) + \alpha\beta E_T \frac{\partial V_1'^{next}(X_1^{next})}{\partial X_1^{next}} \frac{\partial X_1^{next}}{\partial C_T} = 0 \qquad (3.9)$$

Based on Envelope Theorem, $\frac{\partial V_1'^{next}(X_1^{next})}{\partial X_1^{next}} = u'(C_1^{next})$, and according to the budget constraint, $X_1^{next} = Y_1^{next} + R(X_T - C_T)$,

$$\frac{\partial X_1^{next}}{\partial C_T} = -R \qquad (3.10)$$

So the Euler equation for the last period of the household becomes

$$1 = \alpha\beta R E_T (C_1^{next}/C_T)^{-\chi} \qquad (3.11)$$

By dividing all variables by the permanent labor income component (P) again,

$$\frac{C_1^{next}}{C_T} = \frac{C_1^{next}/P_1^{next}}{C_T/P_1^{next}} = \frac{C_1^{next}/P_1^{next}}{C_T/P_T} \frac{P_1^{next}}{P_T} = \frac{c_1^{next}}{c_T} N_1^{next} \qquad (3.12)$$

Here, P_1^{next} is regarded as P_{T+1} as discussed above, so according to chain rule of permanent income P, $\frac{P_1^{next}}{P_T} = N_1^{next}$.

Also, the budget constraint changes to

$$\begin{aligned} x_1^{next} &= \frac{R(X_T - C_T)}{P_T} \frac{P_T}{P_1^{next}} + G_1^{next} V_1^{next} \qquad (3.13) \\ &= \frac{R(x_T - c_T)}{N_1^{next}} + G_1^{next} V_1^{next} \end{aligned}$$

The Euler equation then transforms to:

$$1 = \mathrm{R}\alpha\beta\mathrm{E}_{\mathrm{T}}\left[\left\{\frac{\mathrm{c}_1^{\mathrm{next}}\left[\frac{\mathrm{R}(\mathrm{x}_{\mathrm{T}} - \mathrm{c}_{\mathrm{T}})}{\mathrm{N}_1^{\mathrm{next}}} + \mathrm{G}_1^{\mathrm{next}}\mathrm{V}_1^{\mathrm{next}}\right]\mathrm{N}_1^{\mathrm{next}}}{\mathrm{C}_{\mathrm{T}}}\right\}^{-\chi}\right] \tag{3.14}$$

The idea of numerical solution is as follows: we firstly adopt the numerical result of c_1 obtained from the "without altruism" model (which is a simple backward iteration from c_T to c_1) into Euler equation (3.14) to calculate c_T, secondly, based on this c_T, we use backwards recursive method to compute the optimal consumption in each period. Then using new c_1 as an updated starting point, we iterate the process until it converges.

3.3 Long-run comparisons

3.3.1 Detrended and non-detrended life-cycle consumption

In this paper, due to our model setting, there are two different consumptions: one is the detrended life-cycle consumption $c_{i,t,A}$ and the non-detrended life-cycle consumption $C_{i,t,A}$. To understand how each factor affects the consumption pattern, let us reconsider the Euler equation

$$1 = \mathrm{R}\beta\mathrm{E}\left(\left(\frac{C_{i,t,A}}{C_{i,t-1A-1}}\right)^{-\chi}\right) = \mathrm{R}\beta\mathrm{E}\left(\left(\frac{c_{i,t,A}N_{i,t}}{c_{i,t-1A-1}}\right)^{-\chi}\right) =$$

$$\mathrm{R}\beta\mathrm{E}\left(N_{i,t}{}^{-\chi}\left(\frac{c_{i,t,A}}{c_{i,t-1A-1}}\right)^{-\chi}\right) \tag{3.15}$$

First, we consider a special case with $\mathrm{R}\beta = 1$. Under this case $\frac{c_{i,t,A}}{c_{i,t-1A-1}} = 1$ cannot be a solution since $1 < E(N_{i,t}{}^{-\chi})$. As a result, the detrended consumption must possess an upward tilting trend, i.e. a low consumption level in the young and a high consumption level in the old age. This suppressing consumption at the young age is the reflecting of buffer saving incentive as Carroll (1992) pointed out.

Given the consumption notation, we have

$$C_{i,t,A} = P_{i,t,A} c_{i,t,A}\left(x_{i,t,A}\right) = P_{i,t,A} c_{i,t,A}\left(\frac{Y_{i,t,A} + RX_{i,t-1,A-1}}{P_{i,t,A}}\right) \tag{3.16}$$

In addition, $Y_{i,t,A} = P_{i,t,A}(G_t V_{i,t})$ and $P_{i,t,A} = P_{i,t-1,A} N_{i,t}$. The marginal income propensity to consume with respect to the permanent income shock is $MPC_{permanent} \equiv \partial \log\left(C_{i,t,A}\right) / \partial \log\left(N_{i,t}\right) = 1$ which is consistent with PIH. The marginal propensity to consume with respect to the transitory income shock

$$MPC_{transitory} = \frac{\partial \log(C_{i,t,A})}{\partial \log(V_{i,t})} = \frac{V_{i,t}}{C_{i,t,A}} C'_{i,t,A}(X_{i,t,A}) \tag{3.17}$$

If we concentrate on the long-run wealth profile where $V_{i,t} = 1$ and $C_{i,t,A}$ is on its long-run level, then $MPC_t = \frac{1}{C_{i,t,A}} C'_{i,t,A}(X_{i,t,A})$ which shows the consumption sensitivity to the transitory income at every age in the long run. As we can see, this sensitivity will be affected by the wealth level that the consumer is situated at his age. In the following two sections, we show the MPC results for both finite life model without altruism and finite life model with altruism.

3.3.2 MPC for finite life model without altruism

Figure 11 shows the pattern for finite life-span (FnConst, the solid line) and Infinite (Inf, the dashed line) models. Let's see the infinite case firstly.

Though the initial wealth is zero, for an infinitely living agent, in the long run he will accumulate his wealth to around 0.45 level so that his gross wealth (i.e. income plus the wealth) stays at 1.45. In the long run, he only consumes one unit in each period. The 0.45 will never be consumed as a buffer stock. In this figure, only long-run levels are graphed. To contrast with the case of no buffer saving, we can consider the case without uncertainty. Under this case, Euler equation is $1 = R\beta(\frac{c_{i,t,A}}{c_{i,t-1,A-1}})^{-\chi}$. With $R\beta < 1$ in our calibration, this implies that consumption is decreasing over ages. That is, because of impatience ($\beta R < 1$), people will consume more than his wealth by borrowing the money. Given the agent consumes one unit in each period, simulation shows that in the long run his gross wealth will stay at 0.9398 (not show in the figure), the gap between consumption and gross wealth is his debt. The comparison of the infinite cases with/without uncertainty demonstrates that buffer saving incentive changes the consumption profile completely. Consumption becomes growing with age when household faces uncertainty. Secondly, for a finite horizon model, the agent suppresses his consumption at the young age for buffer incentive. As discussed in chapter 2, uncertainty will decrease consumption and transform consumption into assets as a buffer. A larger uncertainty will lead to a bigger buffer stock.

One major difference that the finite life span causes is that as approaching the end of the life, the agent will try to consume whatever is left, instead of keeping the same consumption level. Meanwhile, gross wealth shows a hump shape in response to the suppression of consumption at the young ages and the acceleration of consumption at the old ages. Another major difference is that long-run marginal propensity of consume (MPC) is not constant any more. MPC in finite case is high at both the ends of the life. Higher MPC at the beginning of life is due to consumption suppression so that the marginal utility of consumption is high. Consumers will take advantage of the high marginal utility gain when there is a good shock, which reflects a large slope at the low gross wealth level in consumption policy functions. On the other hand, by the end of the life, household rationally chooses to consume out of his wealth. Accordingly, consumption goes upwards and wealth moves down, which leads MPC to go up.

[Figure 11: Comparisons of consumption, gross wealth and MPC under finite life-span model and infinite life model]

3.3.3 MPC for finite life model with altruism

Altruism motivation will affect agent's end-of-life consumption decision. Previously, the agent chooses to consume out of his wealth due to his complete selfishness. However, if the agent is altruistic, which seems closer to reality, his optimal end-of-life consumption choice will change to bequeath part of his wealth to the next generation. How much to be left as a bequest depends on his degree of altruism. The higher the degree of altruism is, the more wealth will be transferred to the young and the flatter the consumption and gross wealth lines are. Figure 12 displays the patterns of detrended consumption, gross wealth and MPC under different cases[6]. As we see, as the degree of altruism, α, increases from 0 to 1 (FnConst and Inf could be regarded as $\alpha = 0$ and $\alpha = 1$ respectively), both the ends of these lines become flatter and flatter. This makes sense that since the agent's altruistic motivation becomes stronger, he will consume less and bequeath more at the end of his life so consumption does not go upwards sharply. Correspondingly, gross wealth does not move down and MPC becomes smaller. The more the next generation gets from the old generation, the more gross wealth they will inherit and therefore the more they could consume. So the altruism leads to symmetric changes happened at both the ends of agent's life.

[Figure 12: Comparisons of consumption, gross wealth and MPC under finite life-span model with/without altruism and infinite life model]

3.4 Calibration

In Chapter 2, we calibrate the smoothness ratio and the sensitivity coefficient based on the infinite life model and the income process where

[6] For robust check, we compare finite life-span model with altruism (FnConstantAltru) at $\alpha = 1$ with infinite case because when the agent is complete altruistic and given no income growth, he is expected to behave as though his life is infinite. As shown, the left from old generation (0.45 unit) becomes exactly the buffer-saving required for the young generation in infinite case, and in the long run, the young only consumes one unit in each period.

aggregate shocks are transitory and the household income has a unit root. Results show that in the incomplete information case, its corresponding simulated smoothness ratio is 0.341, lower than in the complete information case and in the real data but its sensitivity coefficient improves to 0.1113, becoming sensitive to the lagged income change and much closer to the reality. The reason behind is that incomplete information causes a misperception of the persistence of a shock which leads to the fact that households do not raise consumption by the amount warranted by the actual persistence of the shock and the relative consumption volatility to income is lowered. Meanwhile, misperception will also create a sluggish response of consumption to the aggregate income shock. This sluggishness of consumption in turn produces a direct correlation between the consumption growth and the lagged income growth.

To increase the smoothness ratio in the incomplete information case, we try on the finite life-span model. There are two reasons why to consider this model. One is because it is much closer to the reality, the other is that from the long-run features comparison above, we figure out that instead of the constant MPC under infinite life model, finite life-span model will create age-varying MPC, in particular, MPC will become larger at both the ends of household's life. And meanwhile, altruism attitude will adjust the pattern of MPC. The higher the degree of altruism is, the flatter the MPC will be. Age-varying MPC is expected to have an impact on consumption volatility.

To be convenient for comparison, we transform the parameter setting to fit quarterly U.S. data again to be consistent with Chapter 2. Meanwhile, due to significant serial correlation in high frequency (quarterly) aggregate shock, and given that household has only incomplete information i.e. observing $(\mathrm{GV})_{\mathrm{i,t}} = \mathrm{G_t V_{i,t}}$ as a whole in each period, household can only project future transitory component on the current one which is expressible as

$$\log\left(\mathrm{G_{t+1} V_{i,t+1}}\right) = \varphi \log\left(\mathrm{G_t V_{i,t}}\right) + \varepsilon_{\mathrm{i,t+1}} \qquad (3.17)$$

where $\varphi = \rho \frac{\sigma_g^2}{\sigma_g^2 + \sigma_v^2}$, $\varepsilon \sim \text{iid } \mathrm{N}(0, (1 - \varphi^2)(\sigma_g^2 + \sigma_v^2))$ and σ_g^2 is the unconditional variance of $\ln\mathrm{G}$.

[Table 9: Parameter setting fit with quarterly frequency]

For calibration, to be consistent with Chapter 2, we simulate 100 samples, each of which consists of a panel of 2000 households across 150 periods again. The first 50 periods are dropped to eliminate the impact of initial wealth condition. Calibration details refer to Appendix 5. Table 10 demonstrates the simulated results under finite life-span model with/without altruism respectively.

[Table 10: Relative smoothness and excess sensitivity: comparisons of infinite life model, finite life-span model with/without altruism]

Compared with the infinite life model as a benchmark, two indications in the finite life span model are indeed improved and closer to the reality. As we can see, the smoothness ratio increases from 0.34 of the benchmark infinite life model to 0.3766 and the sensitivity coefficient increases from 0.1113 to 0.124. That is, age-varying MPC indeed has an influence on the consumption volatility. For altruism attitude, the smoothness ratio is indeed decreasing in comparison with the finite life span model, which is consistent with the age-varying MPC observation above as well. That is, with the degree of altruism increasing, sensitivity of consumption to income declines accordingly. As a result, the relative volatility of consumption to income shifts down as expected.

3.5 Conclusion

All analysis in Chapter 1-2 are based on an infinite horizon model following Deaton (1991) and Ludvigson and Michaelides (2001), which is far from the reality. To be closer to the reality, this chapter extends the model from an infinite horizon to a finite life span. We also introduce the altruism incentive which is a bridge between a finite life span model and an infinite horizon model. In the finite life-span model, marginal propensity to consume becomes age-varying instead of constant. In particular, MPC will become larger at both the ends of a household's life. Altruism attitude will also have an impact on the pattern of MPC. The higher the degree of altruism is, the flatter the MPC will be. For the calibration, compared with the infinite life model, the smoothness ratio and the sensitivity coefficient under a finite life span model are indeed improved and closer to the reality.

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Tables and Figures used in the thesis

Table 1: Ratio of the standard deviation of consumption to the standard deviation of income for AR(1) income with different autocorrelation coefficient, abstracted from Deaton (1991)

AR coeff	-0.4	0.0	0.3	0.5	0.7	0.9
Std(c)/std(y)	**0.43**	**0.5**	**0.67**	**0.67**	**0.78**	**0.94**

Source: Deaton (1991)

Table 2: Unit root test summary

		How Persistent?	
		Permanent (with unit root)	**Transitory (w/o unit root)**
Who Receives?	**Aggregate**		√
	Idiosyncratic	**Some**	√

Table 3: Parameter setting

Preference		Income Process				Other
β	χ	ρ	σ_u	σ_v	σ_n	R
0.99	2	0.85	0.01	0.05	0.05	1

Table 4: Long-run mean level and standard deviation of gross wealth and consumption in complete/incomplete information cases

Detrended		Complete	Incomplete
Gross Wealth	**Mean**	1.1861	1.1544
	Std. Dev.	0.1869	0.1133
Consumption	**Mean**	1.0121	0.8186
	Std. Dev.	0.0298	0.0598

Table 5: Relative smoothness and excess sensitivity: U.S. aggregate quarterly data (1959:Q1—2008:Q4)

	Smoothness Ratio	Sensitivity Coefficient
$\Delta C_t/C_{t-1}$	0.71	0.21
$\Delta C_t^{ND}/C_{t-1}^{ND}$	0.73	0.18
$\Delta C_t^{S}/C_{t-1}^{S}$	0.46	0.09
PIH	1.26	0.00

Real data sources: Bureau of Economic Analysis (BEA)

Table 6: Relative smoothness and excess sensitivity: Ludvigson and Michaelides (2001) simulated results

Information	Smoothness Ratio	Sensitivity Coefficient
Complete	1.09	0.055
	(0.018)	(0.080)
Incomplete	0.91	0.433
	(0.016)	(0.054)

Notes: in parentheses are the standard errors

Table 7: Relative smoothness and excess sensitivity: our model's simulated results

Information	Smoothness Ratio	Sensitivity Coefficient
Complete	0.6374	0.0148
	(0.036)	(0.0715)
Incomplete	0.341	0.1113
	(0.064)	(0.046)

Notes: in parentheses are the standard errors

Table 8: Parameter setting fit with annual frequency

Preference		Income Process			Other
β	χ	σ_g	σ_v	σ_n	R
0.96	2	0.025	0.0968	0.1	1

Table 9: Parameter setting fit with quarterly frequency

Preference		Income Process				Other
β	χ	ρ	σ_u	σ_v	σ_n	R
0.99	2	0.85	0.01	0.05	0.05	1

Table 10: Relative smoothness and excess sensitivity: comparisons of infinite life model, finite life-span model with/without altruism

Incomplete Information	Smoothness Ratio	Sensitivity Coefficient
Infinite life	0.341	0.1113
	(0.064)	(0.046)
Finite Life Span	0.3766	0.124
	(0.0599)	(0.0486)
Altruism (α=0.5)	0.3231	0.119
	(0.0758)	(0.054)

Notes: in parentheses are the standard errors

Figure 1: Log non-asset personal income

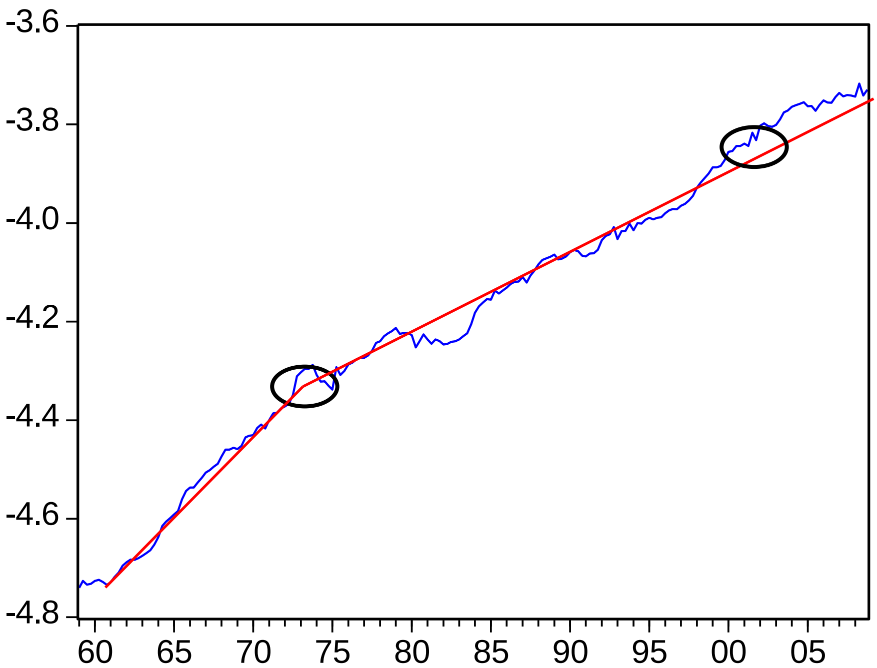


Figure 2: Linear detrended log non-asset personal income

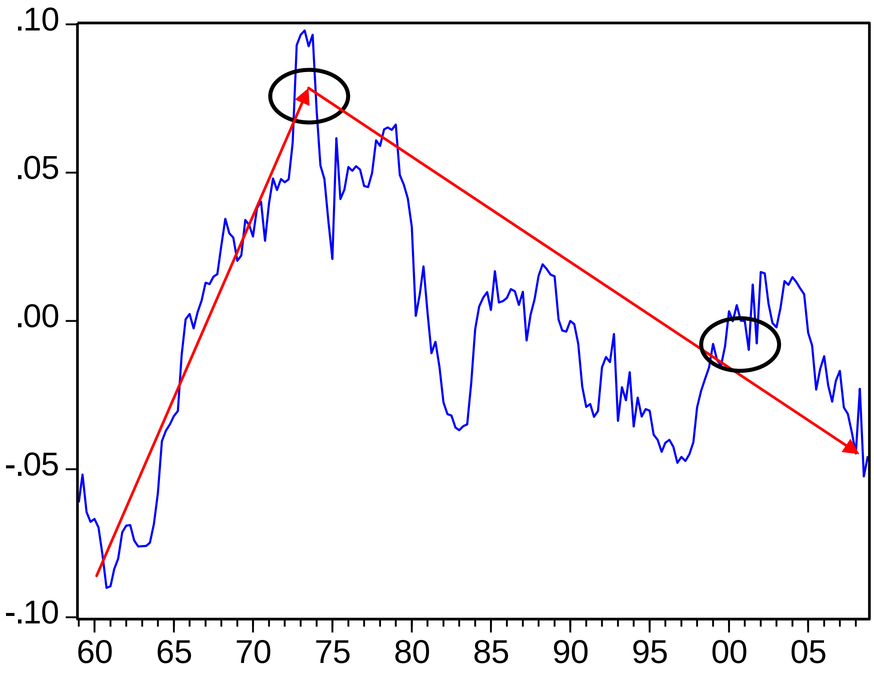


Figure 3: Expected consumption growth

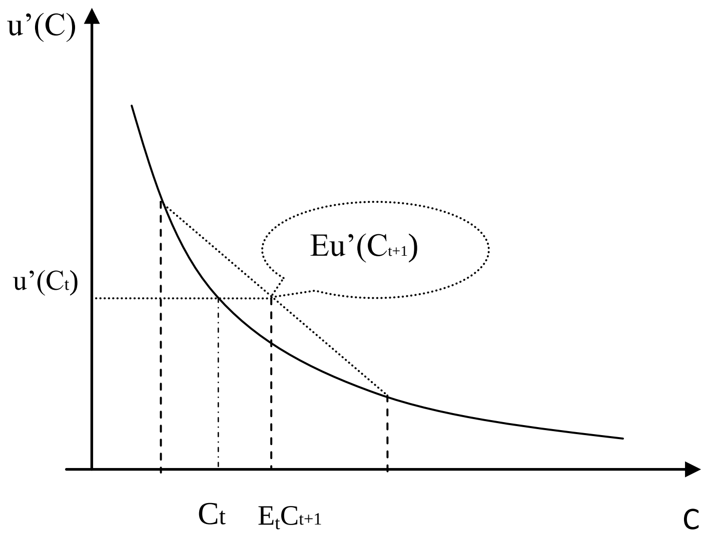


Figure 4: Consumption policy functions of complete/incomplete information

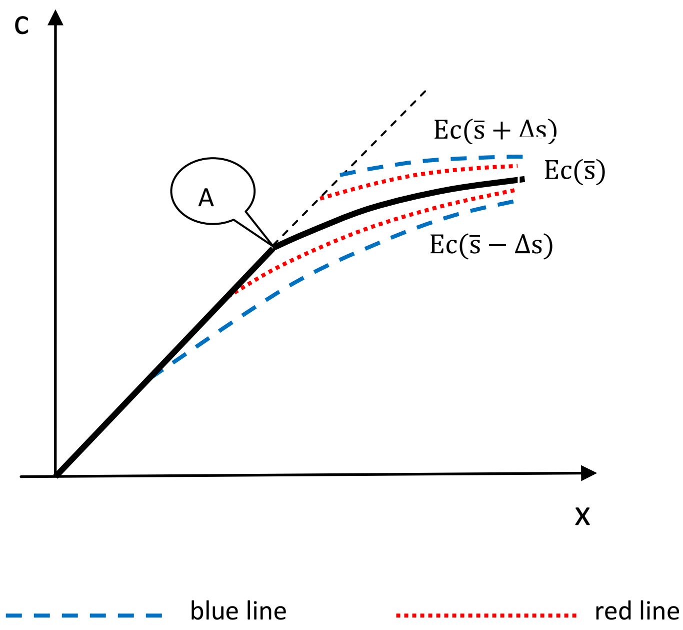
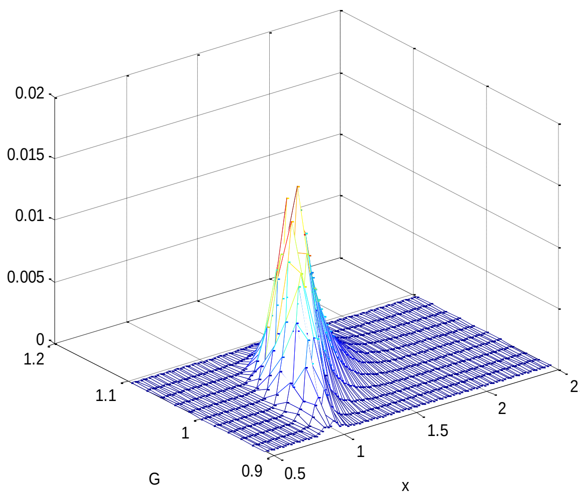
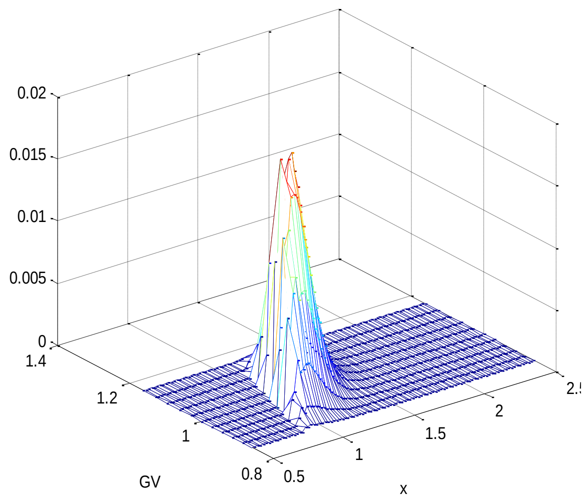


Figure 5: Stationary joint distributions under complete/incomplete information cases

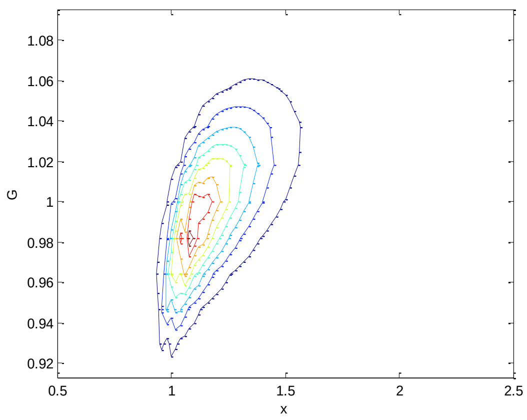


(a) Complete information

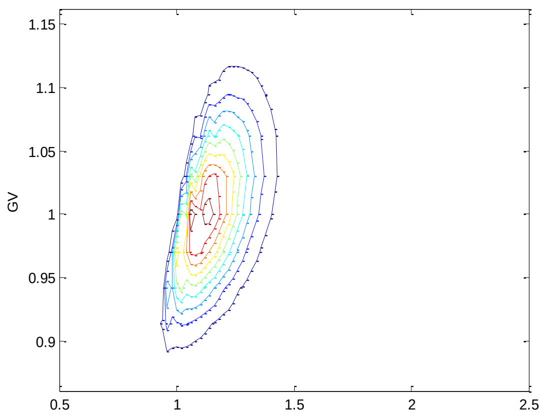


(b) Incomplete information

Figure 6: Contour plots of joint probability under complete/incomplete information cases



(a) Complete information



(b) Incomplete information

Figure 7: Stationary distributions of gross wealth under complete/incomplete information cases

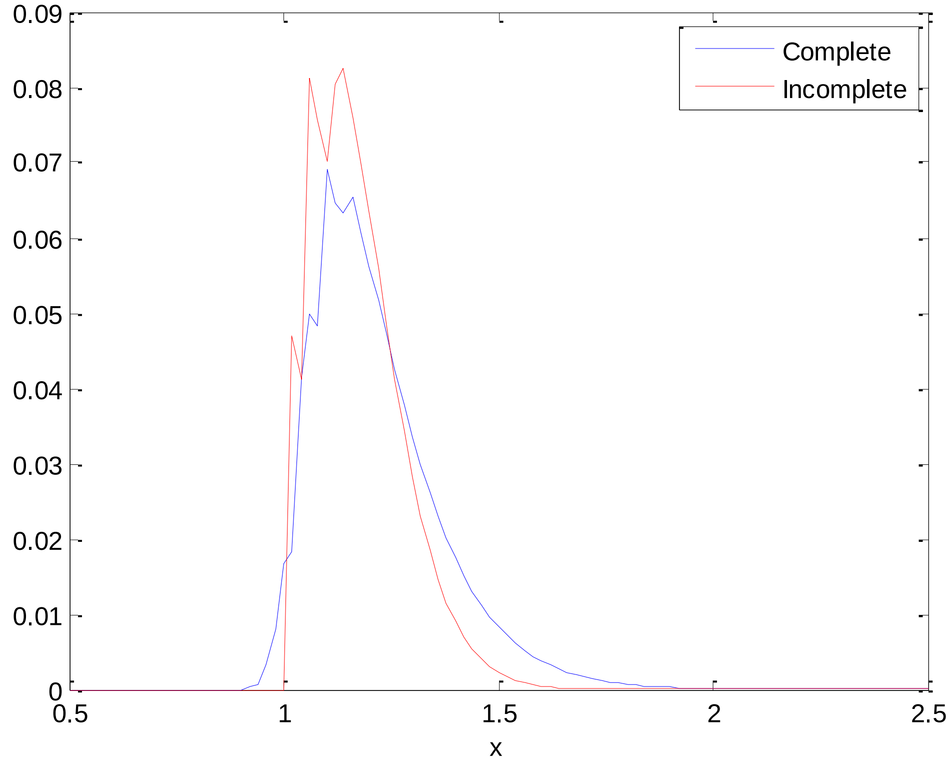


Figure 8: Expected consumption growth rate under precautionary consumption management

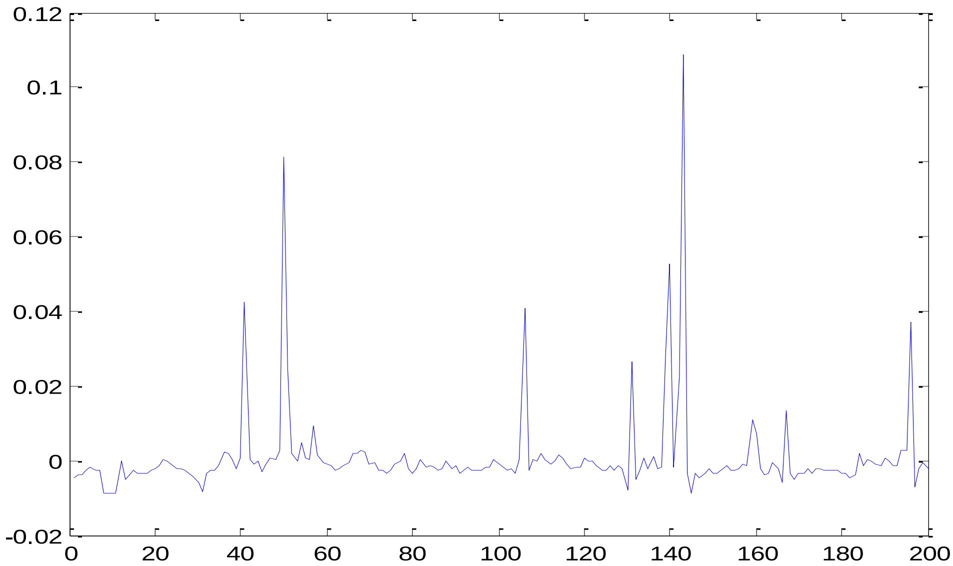


Figure 9: Impulse response to a 1% positive GV shock under complete and incomplete information cases

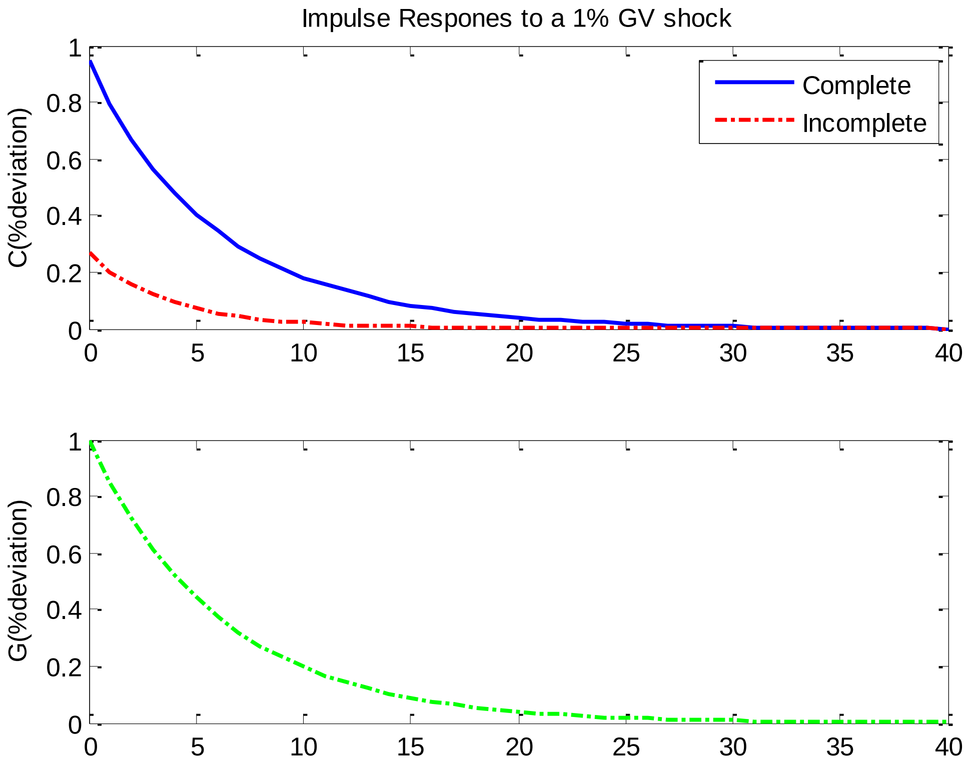


Figure 10: Impulse response to a 1% negative GV shock under complete and incomplete information cases

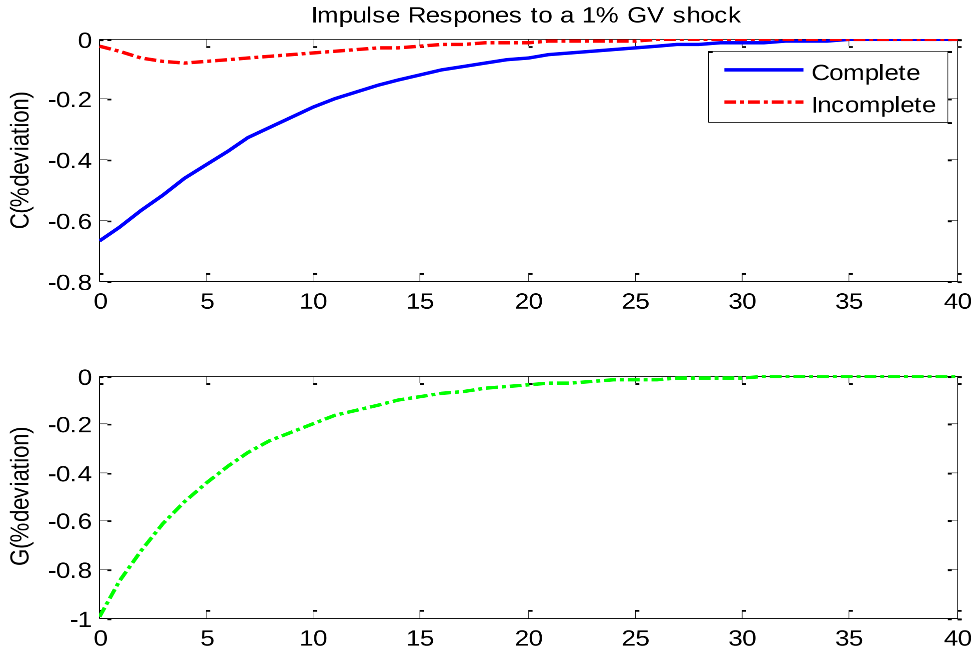


Figure 11: Comparisons of consumption, gross wealth and MPC under finite life-span model and infinite life model

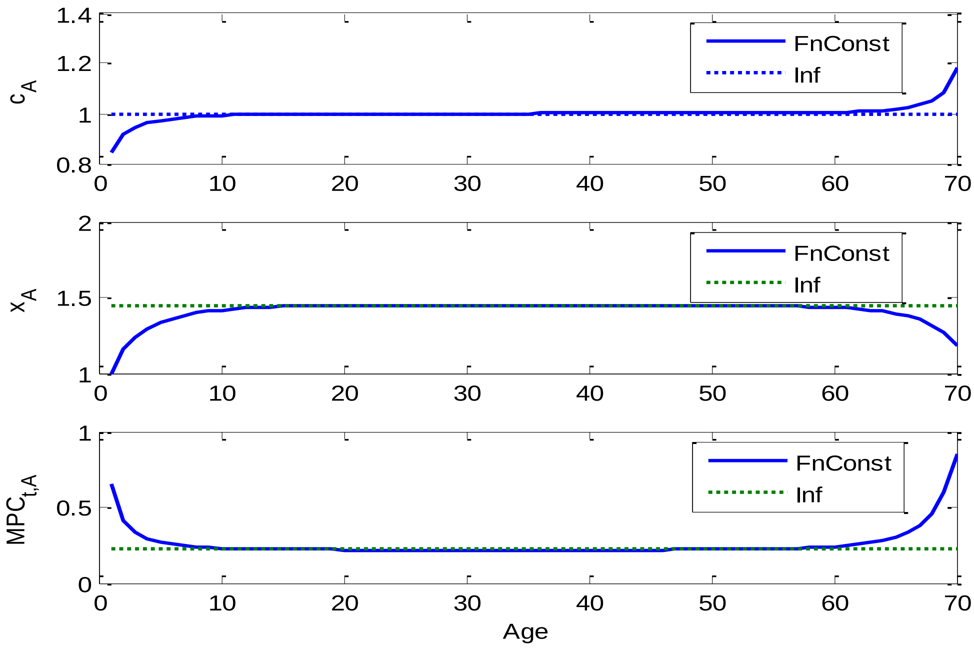
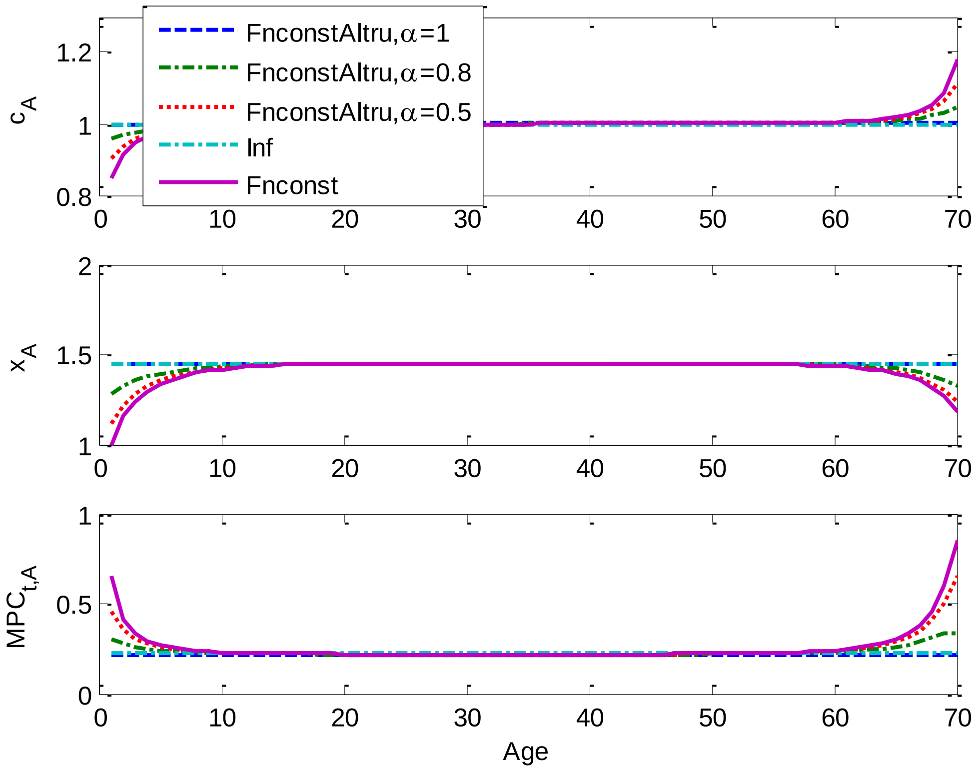


Figure 12: Comparisons of consumption, gross wealth and MPC under finite life-span model with/without altruism and infinite life model



Appendix 1: Construction of shocks

For normal distributed shocks, following Carroll(1997), the lognormal distributions were truncated at three standard deviations from the mean, yielding minimum and maximum values $\underline{V}$, $\underline{N}$, $\overline{V}$, $\overline{N}$. Full numerical integration is extremely slow, so the lognormal distributions were approximated by a ten-point discrete probability distribution. The distance $(\overline{V}-\underline{V})$ was divided evenly into ten regions of size $(\overline{V}-\underline{V})/10$ with individual boundaries denoted as B_j. Associated with each of these regions was the average value of V within the region, computed by calculating the numerical integral $\widehat{V}_j = \int_{B_j}^{B_{j+1}} V dF(V)$. The probability of drawing a shock of value $\widehat{V}_j$ is given by $F(B_{j+1})-F(B_j)$. An analogous procedure was used to approximate the distribution of permanent shock N.

Since a shock state variable (which is G if the information is complete or (GV) if the information is incomplete) follows AR(1) process, following Tauchen (1986), we discretize this shock state properly such that for complete information there is a Markov transition matrix $M = \{m_{ij}\}$ with $m_{ij} = \Pr(G_j|G_i)$ and its implied autocorrelation coefficient being ρ and conditional variance equal to var(u), and for incomplete information, $m_{ij} = \Pr(GV_j|GV_i)$ autocorrelation coefficient ψ as well as conditional variance var(ε).

Appendix 2: Consumption Policy Function under infinite life case

The Bellman's equation for this problem is

$$V(X_t) = \underset{\{C_t\}}{\mathrm{Max}} \{u(C_t) + \beta E_t V(X_{t+1})\}$$

F.O.C. $u'(C_t) + \beta E_t V'(X_{t+1})(-R) = 0$ (A.1)

$$V'(X_t) = \beta R E_t V'(X_{t+1}) \qquad \text{(A.2)}$$

Substitute (A.2) into (A.1), we get

$$V'(X_t) = u'(C_t) \qquad \text{(A.3)}$$

Leading (A.3) one period, and then substituting it into (A.1), we can get Euler Equation as follows:

$$u'(C_{i,t}) = \beta R E_t V'(C_{i,t+1})$$

or, $$1 = R\beta E_t[(C_{i,t+1}/C_{i,t})^{-\chi}] \qquad \text{(A.4)}$$

Dividing both sides by the current level of permanent income

$$\frac{C_{i,t+1}}{C_{i,t}} = \frac{C_{i,t+1}/P_{i,t+1}}{C_{i,t}/P_{i,t+1}} = \frac{C_{i,t+1}/P_{i,t+1}}{C_{i,t}/(P_{i,t}N_{i,t+1})} = \frac{c_{i,t+1}}{c_{i,t}} N_{i,t+1} \qquad \text{(A.5)}$$

Also, the budget constraint changes to

$$\frac{X_{i,t+1}}{P_{i,t+1}} = \frac{R(X_{i,t} - C_{i,t})}{P_{i,t+1}} + \frac{Y_{i,t+1}}{P_{i,t+1}}$$

➔ $$x_{i,t+1} = \frac{R(X_{i,t} - C_{i,t})}{P_{i,t}N_{i,t+1}} + G_{t+1}V_{i,t+1} = \frac{R(x_{i,t} - c_{i,t})}{N_{i,t+1}} + G_{t+1}V_{i,t+1} \qquad \text{(A.6)}$$

The Euler equation then becomes

$$1 = R\beta E_t\left[\left\{\frac{c_{i,t+1}\left[\frac{R(x_{i,t} - c_{i,t})}{N_{i,t+1}} + G_{t+1}V_{i,t+1}\right]N_{i,t+1}}{c_{i,t}}\right\}^{-\chi}\right] \qquad \text{(A.7)}$$

Discretize the random shocks over [0.01 2] with fifty even grids. For a given grid, (X_i,G_j) for complete information or $(X_i,(GV)_j)$ for incomplete information, the way of iteration is to:

For each household h,

Starting from an initial consumption policy function, where we make $c^{(0)} = x$ for all i and j, where i represents the i-th gross wealth state and j represents the j-th income innovation state (i.e. G for the complete information and (GV) for the incomplete information).

1. Look over all possible N and (G, V) for complete information, and N and (GV) for incomplete information at time *t+1* as well as the relevant probabilities Fn, (Fg, Fv) for complete information, and Fn, Fgv for incomplete information to update the k-th round consumption policy $c_{i,j}^{(k)}$ based on the Euler equation:

For complete information

$$R\beta \sum_p \sum_q \sum_l Fg_{j,p} Fn_q Fv_l \left\{ \frac{c^{(k-1)}\left[\frac{R\left(x_i - c_{i,j}^{(k)}\right)}{N_q} + G_p V_l, G_p \right] N_q}{c_{i,j}^{(k)}} \right\}^{-\chi} - 1 = 0 \qquad \text{(A.8)}$$

For incomplete information

$$R\beta \sum_p \sum_q Fgv_p Fn_q \left\{ \frac{c^{(k-1)}\left[\frac{R\left(x_i - c_{i,j}^{(k)}\right)}{N_q} + (GV)_p, (GV)_p \right] N_q}{c_{i,j}^{(k)}} \right\}^{-\chi} - 1 = 0 \qquad \text{(A.9)}$$

We use an interpolation scheme such as cubic splines to interpolate the $c^{(k-1)}$ conditional on $c^{(k)}$, with state variables x_i and G_j for complete information or with state variables x_i and $(GV)_j$ for incomplete information, because the consumption policy function is not actually a function yet. It is a matrix that contains the various values of consumption at each grid point in the discretized x- and G- or GV- space. However, x and G or GV are continuous spaces. The points that do not belong to the grid are not defined by that "matrix". Therefore, we have to interpolate their corresponding values.

To judge whether liquidity constraint is binding or not, we consider

a) If $c_{i,j}^{(k)} = x_i$ but Euler equation still does not hold and is less than 0, it means that liquidity constraint is binding, so $c_{i,j}^{(k)}$ will choose its maximum value equal to x_i.

b) If not, $c_{i,j}^{(k)}$ will be chosen to take the value that makes the Euler equation hold.

2. Iterate $c^{(k)}$, k>=1, until it converges. The convergence criterion used was

$$\text{Max}|c_{i,j}^{(k-1)} - c_{i,j}^{(k)}| < 0.0001$$

Appendix 3: Consumption Policy Function under finite life-span model without Altruism attitude

Discretize the random shocks and the gross wealth space, for a given grid point (X_i,G_j) for complete information or (X_i,$(GV)_j$) for incomplete information, the way of iteration is to

For each household h,

Starting from an initial consumption policy function at household maximum age *A*, where $c = x$ for all i and j, this is because in the last period of life it is optimal to consume everything.

1. Look over all possible N and (G, V) for complete information, and N and (GV) for incomplete information at age *A* as well as the relevant probabilities Fn, (Fg, Fv) for complete information, and Fn and Fgv for incomplete information to compute consumption at *A-1*, $c_{A-1,i,j}$, based on the Euler equation.

   For complete information

$$R\beta \sum_p \sum_q \sum_l Fg_{j,p} Fn_q Fv_l \left\{ \frac{c_A\left[\frac{R\left(x_i - c_{A-1,i,j}\right)}{N_q} + G_p V_l,\, G_p\right] N_q}{c_{A-1,i,j}} \right\}^{-\chi} - 1 = 0 \qquad \text{(A.10)}$$

   For incomplete information

$$R\beta \sum_p \sum_q Fgv_{j,p} Fn_q \left\{ \frac{c_A\left[\frac{R\left(x_i - c_{A-1,i,j}\right)}{N_q} + (GV)_p,\, (GV)_p\right] N_q}{c_{A-1,i,j}} \right\}^{-\chi} - 1 = 0 \qquad \text{(A.11)}$$

We also use an interpolation scheme such as cubic splines to interpolate the c_A conditional on c_{A-1}, with state variables x_i and G_j for complete information or with state variables x_i and $(GV)_j$ for incomplete information.

To judge whether liquidity constraint is binding or not, we consider

a) If $c_{A-1,i,j} = x_i$ but Euler equation still does not hold and is less than 0, it means that liquidity constraint is binding, so $c_{A-1,i,j}$ will choose its maximum value equal to x_i.

b) If not, $c_{A-1,i,j}$ will be chosen to take the value that makes the Euler equation hold.

2. The consumption of age *A-1* household is obtained after step 1. There is no need for further iteration. Similarly, to obtain consumption at age *A-2*, we just replace c_A with c_{A-1}. One round of computation gives us c_{A-2}. We simply continue the computation backward to c_1.

Appendix 4: Calibration under infinite life case

After getting the consumption policy function for infinite life case, we could do the calibration. The procedures are as follows:

We take

t=150 (unit of time, one unit represents one quarter)

Number of households=2000

The sample size for each household=100;

Under each sample

1. Choose long-run mean level of gross wealth (X), permanent income (P) and aggregate shock G for complete information and combination of GV for incomplete information as the initial X, P and G or (GV) respectively;
2. Interpolate initial detrended consumption based on consumption policy function, conditional on initial state variables, X and G for complete information and GV for incomplete information;
3. Randomly draw aggregate shock G for each unit of time;
4. Randomly draw 2000 idiosyncratic shocks N and V for each unit of time;
5. Under each unit of time *t*, calculate gross wealth (detrended) and income for each household *i* as follows:

$$x_{i,t+1} = R(x_{i,t} - c_{i,t})/N_{i,t+1} + G_{t+1}V_{i,t+1}$$

$$P_{i.t} = P_{i,t-1}N_{i,t}$$

$$Y_{i,t} = P_{i,t}(G_t V_{i,t})$$

Here, no matter household has complete or incomplete information, aggregate shock G and idiosyncratic shock V are drawn separately;

6. Interpolate detrended consumption for each household at time t based on consumption policy function, conditional on corresponding state variables, X and G for complete information and GV for incomplete information
7. Calculate non-detrended consumption by taking

$$C_{i,t} = c_{i,t} P_{i.t}$$

8. Sum up the 2000 households' incomes and non-detrended consumptions at time t
9. Drop out first 50 units of time to eliminate the impact of initial wealth condition
10. Calculate the smoothness ratio and sensitivity coefficient
11. Repeat step 1-10 to get another 99 samples of smoothness ratio and sensitivity coefficient
12. Take the sample mean and standard deviation of smoothness ratio and sensitivity coefficient

Appendix 5: Calibration under finite life-span model with/without Altruism attitude

After getting the consumption policy function for infinite life case, we could do the calibration, the procedures are as follows:

We take

T=70*4 (maximal age of household in the society times the number of quarters in each year)

t=150 (unit of time, one unit represents one quarter)

Number of household=2000

The sample size for each household=100;

Under each sample

1. Choose long-run mean level of gross wealth (X) for age 1;
2. Interpolate initial detrended consumption at age 1 based on consumption policy function, conditional on initial state variables, X at age 1 and G for complete information and GV for incomplete information;
3. Update the gross wealth for age 2 by using budget constraint;
4. Iterate the process (step 1-2) to get initial detrended consumption and gross wealth for each age;
5. Choose long-run mean level of income P and Y for every age;
6. Randomly draw aggregate shock G for each unit of time;

7. Randomly draw 2000 idiosyncratic shocks N and V for each unit of time;
8. Under each unit of time t, calculate gross wealth (detrended) and income for each household i at each age A as follows:

$$x_{i,t+1,A+1} = R\left(x_{i,t,A} - c_{i,t,A}\right)/N_{i,t+1} + G_{t+1}V_{i,t+1}$$

$$P_{i,t,A} = P_{i,t-1,A}N_{i,t}$$

$$Y_{i,t,A} = P_{i,t,A}\left(G_t V_{i,t}\right)$$

Here, no matter household has complete or incomplete information, aggregate shock G and idiosyncratic shock V are drawn separately;

9. Interpolate detrended consumption for each household at each age at time *t* based on consumption policy function, conditional on corresponding state variables, X and G for complete information and GV for incomplete information
10. Calculate non-detrended consumption by taking

$$C_{i,t,A} = c_{i,t,A}P_{i.t,A}$$

11. Sum up the 2000 households' incomes and non-detrended consumptions for every age at time *t*
12. Drop out first 50 units of time to eliminate the impact of initial wealth condition
13. Calculate the smoothness ratio and sensitivity coefficient
14. Repeat step 1-10 to get another 99 samples of smoothness ratio and sensitivity coefficient

15. Take the sample mean and standard deviation of smoothness ratio and sensitivity coefficient

Appendix 6: Stationary distribution under complete information

For the one period next in the future, consider the following probability regarding gross wealth X'<x' and income innovation state G'=g'. The *prime* symbol represents the next period. Capital letter represents a random variable and its little case represents a specific outcome and expression is done under the discretized space. The same is in appendix 7.

$$\Pr(X' < x', g') = \Pr\left(\frac{R(x-c)}{N'} + g'v' < x', g'\right) \tag{A.12}$$

$$= \int_{x-c,v'} \Pr\left(\frac{R(x-c)}{N'} + g'v' < x', g' \mid x-c, v'\right) \Pr(x-c, v')$$

$$= \int_{x-c,v'} \Pr\left(\frac{R(x-c)}{N'} + g'v' < x' \mid g', x-c, v'\right) \Pr(g' \mid x-c, v') \Pr(x-c, v')$$

$$= \int_{x-c,v'} \left(1 - \Phi\left(\frac{\ln\left(\frac{R(x-c)}{x' - g'v'}\right)}{\sigma_n}\right)\right) \Pr(g' \mid x-c) \Pr(x-c) \Pr(v')$$

$$= \int_{x-c,v'} \left(1 - \Phi\left(\frac{\ln\left(\frac{R(x-c)}{x' - g'v'}\right)}{\sigma_n}\right)\right) \Pr(g', x-c) \Pr(v')$$

$$= \int_{x-c} \left[\int_{v'} \left(1 - \Phi\left(\frac{\ln\left(\frac{R(x-c)}{x' - g'v'}\right)}{\sigma_n}\right)\right) \Pr(v')\right] \Pr(g', x-c)$$

$$= \Pr(g') - \int_{x-c} \left[\int_{v'} \Phi\left(\frac{\ln\left(\frac{R(x-c)}{x' - g'v'}\right)}{\sigma_n}\right) \Pr(v')\right] \Pr(g', x-c)$$

The fourth line uses the fact that X-C and V' are independent. Pr(v') is specified according to its lognormal distribution. For Pr(g',x-c) and Pr(x-c),

$$\Pr(g', x-c) = \int_{g} \Pr(g', g, x-c)$$

$$= \int_{g} \Pr(g' \mid g, x-c) \Pr(g, x-c) = \int_{g} \Pr(g' \mid g) \Pr(g, x-c) \tag{A.13}$$

The last line uses the fact that G is a sufficient statistics for G'.

The numerical procedure is starting from an initial $\Pr(g, x - c)$ to compute (A.13), then update $\Pr(g, x - c)$ by using (A.12). The procedure is iterated until it converges.

Appendix 7: Stationary distribution under incomplete information

Consider the following probability regarding gross wealth X<x and income innovation state GV=η,

$\Pr(X' < x', (GV)' = \eta')$

$= \int_{x-c} \Pr(X' < x' | X - C = x - c, (GV)' = \eta') \Pr(X - C = x - c, (GV)' = \eta')$

$= \int_{x-c} \Pr(\frac{R(x-c)}{N'} + \eta' < x') \Pr(X - C = x - c, (GV)' = \eta')$

$= \int_{x-c} \Pr(N' > \frac{R(x-c)}{x' - \eta'}) \Pr(X - C = x - c, (GV)' = \eta')$

$= \int_{x-c} [1 - \Phi\left(\frac{\ln\left(\frac{R(x-c)}{x' - \eta'}\right)}{\sigma_n}\right)] \Pr(X - C = x - c, (GV)' = \eta')$

$= \Pr(\eta') - \int_{x-c} \Phi\left(\frac{\ln\left(\frac{R(x-c)}{x' - \eta'}\right)}{\sigma_n}\right) \Pr(X - C = x - c, (GV)' = \eta')$

$= \Pr(\eta') - \int_{x-c} \Phi\left(\frac{\ln\left(\frac{R(x-c)}{x' - \eta'}\right)}{\sigma_n}\right) \int_{\eta} \Pr((GV)' = \eta' | GV = \eta) \Pr(X - C = x - c, GV = \eta)$ (A.14)

The last line is based on the fact that

$\Pr(X - C, (GV)') = \int_{\eta} \Pr((GV)' | GV = \eta, X - C) \Pr(X - C = x - c, GV = \eta)$

and that $\Pr((GV)' | GV, X - C) = \Pr((GV)' | GV)$ since GV is Markovian.

The numerical procedure is starting from an initial $\Pr(X - C, GV)$ to compute $\Pr(X - C, (GV)')$, then update $\Pr(X - C, GV)$ by using $\Pr(X - C, (GV)')$.

The procedure is iterated until it converges.